47. On the Curves Developable on Two-dimensional Spheres in the Conformally Connected Manifold.

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In two previous papers[1], we have obtained the following Frenet formulae:

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(1)\qquad \left\{
\begin{aligned}
&\frac{d}{d\sigma}\underset{(0)}{A}=\underset{(1)}{A}\,,\\
&\frac{d}{d\sigma}\underset{(1)}{A}=\lambda\underset{(0)}{A}+\underset{(2)}{A}\,,\\
&\frac{d}{d\sigma}\underset{(2)}{A}=\lambda\underset{(1)}{A}+\underset{(3)}{A}\,,\\
&\frac{d}{d\sigma}\underset{(3)}{A}=\underset{(0)}{A}+\overset{4}{\lambda}\underset{(4)}{A}\,,\\
&\frac{d}{d\sigma}\underset{(4)}{A}=-\overset{4}{\lambda}\underset{(3)}{A}+\overset{5}{\lambda}\underset{(5)}{A}\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{d}{d\sigma}\underset{(\infty)}{A}=-\overset{\infty}{\lambda}\underset{(n)}{A}\,,
\end{aligned}
\right.
$$

for the curves in the conformally connected manifold, and have shown that the curve for which $\lambda=\overset{4}{\lambda}=\cdots=\overset{n}{\lambda}=0$ is a generalized loxodrome which cuts all the circles passing through two fixed points always by the fixed angle $\frac{\pi}{4}$, and the curves for which $\lambda=\text{const.}$ and $\overset{4}{\lambda}=\cdots=\overset{\infty}{\lambda}=0$ are the generalized loxodromes which cuts all such circles by the fixed angle $\frac{\pi}{4}-\frac{\varphi}{2}$, where $\lambda=\tan\varphi$.

In the present paper, we shall deal with the curves for which $\overset{4}{\lambda}=\cdots=\overset{\infty}{\lambda}=0$, λ being in general the function of the conformal arc length σ. In this case, if we develop our curve on the tangent conformal space at a point of the curve, its development will be on a two-dimensional sphere, thus, we can treat the curve as if it were in a two-dimensional flat conformal space.

The main theorem which we propose to prove in this paper is the following:

Theorem: If we take a circle which cuts the curve by a certain

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1) K. Yano and Y. Mutô: On the conformal arc length, Proc. **17** (1941), 318–322, On the generalized loxodromes in the conformally connected manifold, Proc. **17** (1941), 455–460.

angle α $\left(0 \leqq \alpha \leqq \frac{\pi}{2}\right)$ *and assume that this circle and its four consecutive circles, which cut the curve also by the same angle* α *and not cut mutually, belong to a coaxal system, then we must have, at the point,*

$$\tan 2\alpha = -\frac{1}{\lambda}$$

the limiting points P and Q of the system being given by

$$P = \frac{\sec^2 \alpha}{2\sqrt{\tan \alpha}} \underset{(0)}{A} + \tan \alpha \underset{(1)}{A} + \sqrt{\tan \alpha}\, \underset{(2)}{A} + \underset{(3)}{A},$$

$$Q = -\frac{\sec^2 \alpha}{2\sqrt{\tan \alpha}} \underset{(0)}{A} + \tan \alpha \underset{(1)}{A} - \sqrt{\tan \alpha}\, \underset{(2)}{A} + \underset{(3)}{A}.$$

The point $\underset{(2)}{A}$ is the intersection of the osculating circle of the curve and the circle $\underset{(1)}{A} - \tan \alpha \underset{(3)}{A}$ which cuts all the circles of the system orthogonally and passes through four points P, Q, $\underset{(0)}{A}$ and $\underset{(2)}{A}$.

Proof. The circle K which cuts the curve by a fixed angle α $\left(0 \leqq \alpha \leqq \frac{\pi}{2}\right)$ may be expressed as follows:

$$K = f(\sigma) \underset{(0)}{A} + \tan \alpha \underset{(1)}{A} + \underset{(3)}{A} \tag{2}$$

where $f(\sigma)$ is a function of the conformal arc length σ.

Differentiating the circle K with respect to σ, and taking account of the Frenet formulae, we have successively

$$K' = (f' + \lambda \tan \alpha + 1) \underset{(0)}{A} + f \underset{(1)}{A} + \tan \alpha \underset{(2)}{A}, \tag{3}$$

$$K'' = (f'' + \lambda' \tan \alpha + \lambda f) \underset{(0)}{A} + (2f' + 2\lambda \tan \alpha + 1) \underset{(1)}{A} + f \underset{(2)}{A} + \tan \alpha \underset{(3)}{A}, \tag{4}$$

$$\begin{aligned} K''' = {} & (f''' + \lambda'' \tan \alpha + \lambda' f + 3\lambda f' + 2\lambda^2 \tan \alpha + \lambda + \tan \alpha) \underset{(0)}{A} \\ & + (3f'' + 3\lambda' \tan \alpha + 2\lambda f) \underset{(1)}{A} + (3f' + 2\lambda \tan \alpha + 1) \underset{(2)}{A} + f \underset{(3)}{A}, \end{aligned} \tag{5}$$

$$\begin{aligned} K'''' = {} & (f'''' + \lambda''' \tan \alpha + \lambda'' f + 4\lambda' f' + 6\lambda f'' \\ & + 7\lambda\lambda' \tan \alpha + \lambda' + 2\lambda^2 f + f) \underset{(0)}{A} \\ & + (4f''' + 4\lambda'' \tan \alpha + 3\lambda' f + 8\lambda f' + 4\lambda^2 \tan \alpha + 2\lambda + \tan \alpha) \underset{(1)}{A} \\ & + (6f'' + 5\lambda' \tan \alpha + 2\lambda f) \underset{(2)}{A} + (4f' + 2\lambda \tan \alpha + 1) \underset{(3)}{A}, \end{aligned} \tag{6}$$

where dushes denote the differentiation with respect to σ.

If, at a point of the curve, the five consecutive circles belong to a coaxal system, then the circles K'', K''' and K'''' must be the linear combinations of the two circles K and K'.

Thus, expressing the condition that K'' is a linear combination of K and K', we have

(7) $f''+\lambda'\tan\alpha+\lambda f=f\tan\alpha+f\cot\alpha(f'+\lambda\tan\alpha+1)$,

(8) $2f'+2\lambda\tan\alpha+1=\tan^2\alpha+f^2\cot\alpha$.

Similarly, the fact that K''' is a linear combination of K and K' gives us

(9) $$f'''+\lambda''\tan\alpha+\lambda' f+3\lambda f'+2\lambda^2\tan\alpha+\lambda+\tan\alpha$$
$$=f^2+\cot\alpha(3f'+2\lambda\tan\alpha+1)(f'+\lambda\tan\alpha+1),$$

(10) $3f''+3\lambda'\tan\alpha+2\lambda f=f\tan\alpha+f\cot\alpha(3f'+2\lambda\tan\alpha+1)$.

Finally, K'''' being also a linear combination of K and K', we find

(11) $$f''''+\lambda'''\tan\alpha+\lambda''f+4\lambda'f'+6\lambda f''+7\lambda\lambda'\tan\alpha+\lambda'+2\lambda^2f+f$$
$$=f(4f'+2\lambda\tan\alpha+1)+\cot\alpha(6f''+5\lambda'\tan\alpha+2\lambda f)\times$$
$$(f'+\lambda\tan\alpha+1),$$

(12) $$4f'''+4\lambda''\tan\alpha+3\lambda'f+8\lambda f'+4\lambda^2\tan\alpha+2\lambda+\tan\alpha$$
$$=\tan\alpha(4f'+2\lambda\tan\alpha+1)+f\cot\alpha(6f''+5\lambda'\tan\alpha+2\lambda f).$$

Now, forming the equation (7) × 3–(10), we have

$$\lambda f=2f\tan\alpha+\lambda f+2f\cot\alpha,$$

$$2f(\tan\alpha+\cot\alpha)=0,$$

from which

(13) $$f=0.$$

Substituting this value of f in the equation (8), we have

(14) $$f'=\frac{1}{2}(\tan^2\alpha-2\lambda\tan\alpha-1).$$

If $f=0$ is substituted in (7) or (10), we find

(15) $$f''=-\lambda'\tan\alpha.$$

If we substitute the value of f, f' and f'' in (9) and (12), we have respectively

(16) $$4f'''=-4\lambda''\tan\alpha-8\lambda\tan^2\alpha+4\lambda^2\tan\alpha-4\tan\alpha$$
$$+3\sec^2\alpha\tan\alpha-\cot\alpha\sec^2\alpha$$

and

(17) $$4f'''=-4\lambda''\tan\alpha-6\lambda\tan^2\alpha+4\lambda^2\tan\alpha-2\tan\alpha+2\lambda+2\tan^3\alpha.$$

From these equations, we find

(18) $$\lambda=\frac{1}{2}\,(\tan\alpha-\cot\alpha)$$

or

(19) $$\tan 2\alpha=-\frac{1}{\lambda}\,.$$

This gives the geometrical meaning of the conformal curvature λ. The equation (11) gives the value of f''''.

The values of f, f', f'', f''' and f'''' being thus determined, the K and K' become

(20) $$K=\tan\alpha \underset{(1)}{A}+\underset{(3)}{A}\,,$$

(21) $$K'=\frac{1}{2}\sec^2\alpha \underset{(0)}{A}+\tan\alpha \underset{(2)}{A}\,,$$

where

(22) $$K^2=\sec^2\alpha \quad\text{and}\quad K'^2=-\tan\alpha\sec^2\alpha\,,$$

consequently the K' is an imaginary circle, and the circles of the coaxal system cut orthogonally all the circles passing through two fixed points.

We shall find these points. They must have the form

$$K+aK'\,,\qquad (K+aK')^2=0\,,$$

from which

$$K^2+2aKK'+a^2K'^2=0\,,$$

$$\sec^2\alpha+a^2(-\tan\alpha\sec^2\alpha)=0\,,$$

$$a=\pm\frac{1}{\sqrt{\tan\alpha}}\,.$$

Consequently, denoting these points by P and Q, we have

(23) $$P=\frac{\sec^2\alpha}{2\sqrt{\tan\alpha}}\underset{(0)}{A}+\tan\alpha\underset{(1)}{A}+\sqrt{\tan\alpha}\,\underset{(2)}{A}+\underset{(3)}{A}\,,$$

(24) $$Q=-\frac{\sec^2\alpha}{2\sqrt{\tan\alpha}}\underset{(0)}{A}+\tan\alpha\underset{(1)}{A}-\sqrt{\tan\alpha}\,\underset{(2)}{A}+\underset{(3)}{A}\,.$$

Finally from the equations

$$(\underset{(1)}{A}-\tan\alpha\underset{(3)}{A})K=0 \quad\text{and}\quad (\underset{(1)}{A}-\tan\alpha\underset{(3)}{A})K'=0\,,$$

we can conclude that the circle $\underset{(1)}{A}-\tan\alpha\underset{(3)}{A}$ passing through P, Q, $\underset{(0)}{A}$ and $\underset{(2)}{A}$ cut orthogonally all the circles of the coaxal system and that $\underset{(2)}{A}$ is the intersection of this circle and the osculating circle of the curve.

Thus the theorem is completely proved.

If we suppose that the conformal curvature λ is constant along the curve, and put

$$\lambda=\tan\varphi \qquad \left(-\frac{\pi}{2}\leqq\varphi\leqq\frac{\pi}{2}\right)$$

then the equation (19) shows us that our angle α is also constant along the curve and gives us

$$\tan 2\alpha=-\cot\varphi$$
$$=\tan\left(\frac{\pi}{2}+\varphi\right)$$

from which

$$\alpha=\frac{\pi}{4}+\frac{\varphi}{2}.$$

Thus α being constant, we have

$$\frac{d}{d\sigma}P=\frac{\sec^2\alpha}{2\sqrt{\tan\alpha}}\underset{(1)}{A}+\tan\alpha(\tan\varphi\underset{(0)}{A}+\underset{(2)}{A})+\sqrt{\tan\alpha}(\tan\varphi\underset{(1)}{A}+\underset{(3)}{A})+\underset{(0)}{A}$$
$$=\frac{\sec^2\alpha}{2\sqrt{\tan\alpha}}\underset{(1)}{A}+\tan\alpha\left[\frac{\tan^2\alpha-1}{2\tan\alpha}\underset{(0)}{A}+\underset{(2)}{A}\right]$$
$$+\sqrt{\tan\alpha}\left[\frac{\tan^2\alpha-1}{2\tan\alpha}\underset{(1)}{A}+\underset{(3)}{A}\right]+\underset{(0)}{A}$$
$$=\frac{\sec^2\alpha}{2}\underset{(0)}{A}+\frac{2\tan^2\alpha}{2\sqrt{\tan\alpha}}\underset{(1)}{A}+\tan\alpha\underset{(2)}{A}+\sqrt{\tan\alpha}\underset{(3)}{A}=\sqrt{\tan\alpha}P,$$
$$\frac{d}{d\sigma}Q=-\sqrt{\tan\alpha}Q,$$

hence, the points P and Q are fixed, and the angle θ between the curve and the circle $\underset{(1)}{A}-\tan\alpha\underset{(3)}{A}$ is given by

$$\sin\theta=\frac{A(\underset{(1)}{A}\underset{(1)}{\ }-\tan\alpha\underset{(3)}{A})}{\sqrt{\underset{(1)}{A}\underset{(1)}{A}}\sqrt{(\underset{(1)}{A}-\tan\alpha\underset{(3)}{A})(\underset{(1)}{A}-\tan\alpha\underset{(3)}{A})}}=\frac{1}{\sec\alpha}=\cos\alpha$$

or

$$\theta=\frac{\pi}{2}-\alpha=\frac{\pi}{4}-\frac{\varphi}{2}, \tag{26}$$

and we see that the curve is a generalized loxodrome as is already shown[1)].

1) K. Yano and Y. Mutô: On the generalized loxodromes in the conformally connected manifold, loc. cit.