107. *An Abstract Integral* (X).

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Introduction. The first section is devoted to simplify the theory of general Denjoy integral. The essential point is to use Romanowski's lemma[1]. He used the lemma to develop the theory of the special Denjoy integral in abstract space. In § 2 we define an "abstract Denjoy integral" The integral which is called $\mathfrak{D}$-integral, becomes general or special Denjoy integral and others by the suitable specialization. The ($\mathfrak{D}$)-integral is defined as the inverse of an "abstract derivative "$\mathfrak{A}\mathfrak{D}$ which is defined axiomatically. Finally, we remark that the theory developed here can be extended to the case of abstract valued functions defined in an abstract space.

§ **1.** Let $f(x)$ be a real valued function in the interval $I_0=(a, b)$. If $f(x)$ is a continuous function in I_0 such that there is a sequence of sets (E_k) such as $I_0=\bigvee_{k=1}^{\infty} E_k$ and $f(x)$ is absolutely continuous in E_k $(k=1, 2, 3, \dots)$, then $f(x)$ is called to be generalized absolutely continuous in I_0, and we write $f\varepsilon CAC_{I_0}$ or simply $f\varepsilon GAC$. Approximate derivative $ADF(x)$ of $f(x)$ is defined in the ordinary manner.

We will begin by two lemmas.

(1.1) Let E be a closed set in I_0 and $I_0=\bigvee_{k=1}^{\infty} E_k$, then there is a portion P of E such that a suitable E_k is dense in P.

Proof. If the theorem is not true, then there is a portion P_1 of E such that $P_1 \frown E_1=\theta$. There is also a portion P_2 of P_1 such that $P_2 \frown E_2=\theta$. Thus proceeding we get a sequence (P_k) of portions such that $P_k \geqq P_{k+1}$ $(k=1, 2, 3, \dots)$. Evidently $\bigwedge_{k=1}^{\infty} P_k \neq \theta$. If $x\varepsilon \bigwedge_{k=1}^{\infty} P_k$, then $x\varepsilon E$. On the other hand $x\bar{\varepsilon} E_k$ $(k=1, 2, 3, \dots)$, and then $x\bar{\varepsilon} I_0$ which is a contradiction.

(1.2) (Romanowski) Let $\mathfrak{J}$ be a system of open intervals in I_0, such that[1]

1°. $I_k\varepsilon\mathfrak{J}$ $(k=1, 2, \dots, n)$ and $(\bigvee_{k=1}^{\infty} \bar{I}_k)^0=I$ imply $I\varepsilon\mathfrak{J}$.

2°. $I\varepsilon\mathfrak{J}$ and $\mathfrak{J}' \leqq I$ imply $I'\varepsilon\mathfrak{J}$.

3°. if $\bar{I}' \leqq I$ implies $I'\varepsilon\mathfrak{J}$, then $I\varepsilon\mathfrak{J}$.

4°. if I_1 is a subsystem of $\mathfrak{J}$ such that $\mathfrak{J}_1$ does not cover I_0, then there is an $I\varepsilon\mathfrak{J}$ such that $\mathfrak{J}_1$ does not cover I.

Then $I_0\varepsilon\mathfrak{J}$.

Proof. 4° implies $\bigvee(I;\ I\varepsilon\mathfrak{J}) \geqq I_0$. Let $\bar{I} \subset I_0$. By the Heine-Borel theorem there are I_k $(k=1, 2, \dots, n)$ in $\mathfrak{J}$ such as $I \subset \bigvee_{k=1}^{n} I_k$. End

1) Romanowski, Recueil math., 1940.

points of I and I_k $(k=1, 2, \dots, n)$ devides I_0 in a system of open intervals $(J_k;\ k=1, 2, \dots, m)$. $J_k \subset I$ implies the existence of k' such that $J_k \subset I_{k'}$ By 2° $J_k \varepsilon \mathfrak{J}$. Thus 2° implies $I \varepsilon \mathfrak{J}$. Finally by 3° $I_0 \varepsilon \mathfrak{J}$.

1.1. We will now prove fundamental theorems. They are well known but the proof given here is very simple.

(1.3) If $f \varepsilon GAC_{I_0}$ and $ADf(x)=0$ almost everywhere in I_0, then $f(x)$ is constant.

Proof. Let I be a system of all intervals in which f is constant. $\mathfrak{J}$ satisfies evidently 1°, 2° and 3° in (1.2). If we show that 4° is satisfied, then the theorem is proved by (1.2). Let $\mathfrak{J}_1$ be a subsystem of $\mathfrak{J}$ and E be a set not covered by $\mathfrak{J}_1$. By $f \varepsilon GAC$ there is a sequence of sets (E_k) such that $\bigvee_{k=1}^{\infty} E_k = I_0$ and $f(x)$ is absolutely continuous in E_k $(k=1, 2, 3. \dots)$. By (1.1) there is a portion[1] $P=[\alpha, \beta] \frown E$ such that $\bar{E}_k \supset P$ for a suitable k. Since f is continuous in I_0, f is absolutely continuous in P. Thus $f'(x)$ exists and $=ADf(x)$ almost everywhere. By $ADf(x)=0$ almost everywhere in I_0, f is constant in $[\alpha, \beta]$. Therefore $[\alpha, \beta] \varepsilon \mathfrak{J}$ but is not covered by $\mathfrak{J}_1$.

(1.4) If $f \varepsilon GAC$, then $f(x)$ is infinitely approximately differentiable almost everywhere.

Proof. Let $\mathfrak{J}$ be a system of intervals in which $f(x)$ is finitely approximately differentiable almost everywhere. $\mathfrak{J}$ satisfies 1°, 2°, and 3° in (1.2). In orddr to prove 4° we proceed as in (1.3). Take P as in (1.3), then f is absolutely continuous in P. Therefore f is finitely differentiable almost everywhere in P. That is, f is finitely approximately differentiable almost everywhere in $[\alpha, \beta]$. Thus $[\alpha, \beta] \varepsilon \mathfrak{J}$ but is not covered by $\mathfrak{J}_1$.

1.2. Under these preparations we can state the definition of the general Denjoy integral as follows:

(1.5) $f(x)$ is integrable in the general Denjoy sense if there is a function $F(x) \varepsilon GAC$ such that $ADF(x)=f(x)$ almost everywhere.

Uniqueness of $F(x)$ follows from (1.4).

§ **2.** We will now generalize the above integral.

(2.1). If $f(x)$ is a real valued function defined on $I=(a, b)$, then $f(x)$ is called absolutely continuous in $E \subset I$ and is written $f \varepsilon AC_E$ provided that

1°. $E_1 \subset E$ and $f \varepsilon \mathfrak{AC}_E$ imply $f \varepsilon \mathfrak{AC}_{E_1}$,

2°. if $E \subset (a, b)$, $f(x)$ is continuous in (a, b) and $f \varepsilon \mathfrak{AC}_E$ then $f \varepsilon \mathfrak{AC}_{\bar{E}}$,

and an operation $\mathfrak{AD}f(x)$ (abstract derivative of $f(x)$ at x) is defined and

3°. For a closed set F, $f \varepsilon \mathfrak{AC}_F$ implies the almost everywhere existence of $\mathfrak{AD}f(x)$.

4°. if $f(x)$ is continuous in $I=(a, b)$ and is locally constant in the complement of a closed set F in I, and further if $f \varepsilon \mathfrak{AC}_F$ and $\mathfrak{AD}f(x)=0$ almost everywhere in F, then $f(x)$ is constant in I.

(2.2) $f(x)$ is called generalized absolutely continuous in I_0 and we

1) $[\alpha, \beta]$ denotes closed interval.

write $f \epsilon \mathfrak{GAC}_{I_0}$, if there is a sequence of sets (E_k) such that $I_0 = \bigvee_{k=1}^{\infty} E_k$ and $f \epsilon \mathfrak{AC}_{E_K}$ $(k=1, 2, 3, \ldots)$.

By (2.1′), (2.2′) we denote (2.1), (2.2) where E, E_1 and E_k are supposed closed, and then 2° becomes evident. In this case we denote $\mathfrak{GAC}$ by $\mathfrak{GAC}'$.

We can prove that

(2.3) If $f \epsilon \mathfrak{GAC}_{I_0}$ and $\mathfrak{AD}f(x)=0$ almost everywhere in I_0, then $f(x)$ is constant in I_0.

(2.4) If $f \epsilon \mathfrak{GAC}_{I_0}$, then there exists $\mathfrak{AD}f(x)$ almost everywhere in I_0.

Thus we can define the integral

(2.5) $f(x)$ is called $\mathfrak{D}$-integrable if there is a function $F(x) \epsilon \mathfrak{GAC}$ such that $\mathfrak{AD}F(x)=f(x)$ almost everywhere. $F(x)$ is denoted by $(\mathfrak{D})\int_I f(t)dt$.

2.1. If we suppose that

(2.6) $\mathfrak{AC}$ and $\mathfrak{AD}$ satisfy the condition:

5°. $\mathfrak{AD}$ is an additive operation, that is if $\mathfrak{AD}f(x)$ and $\mathfrak{AD}g(x)$ exist, then $\mathfrak{AD}\big(af(x)+bg(x)\big)$ exists and is equal to $a\mathfrak{AD}f(x)+b\mathfrak{AD}g(x)$, a and b being constant,

6°. for a closed set F in an interval I, if $f \epsilon \mathfrak{AC}_F$, $\mathfrak{AD}f(x) \geqq 0$ almost everywhere in F and $f(x)$ is $\leqq 0$ in $I-F$, then $f(x) \geqq 0$ in I.

Then we have

(2.7) If $f(t)$ and $g(t)$ are $(\mathfrak{D})$-integrable in (a, b), then $af(t)+bg(t)$ is also, and $a(\mathfrak{D})\int_a^b f(t)dt + b(\mathfrak{D})\int_a^b g(t)dt = (\mathfrak{D})\int_a^b \big(af(t)+bg(t)\big)dt$.

(2.8) If $f(x)$ is $(\mathfrak{D})$-integrable and $f(x)$ is $\geqq 0$ almost everywhere, then $f(x)$ is integrable in the Lebesgue sense.

Proof. It is sufficient to prove that $(\mathfrak{D})\int_a^b f(t)dt \geqq 0$, which follows from (1.2) and the definition of the $(\mathfrak{D})$-integral

(2.9) Let $\big(f_n(t)\big)$ be a convergent sequence of $(\mathfrak{D})$-integrable functions in I such that $g(x) \leqq f_n(x) \leqq h(x)$ $(n=1, 2, 3, \ldots)$, $g(x)$ and $h(x)$ being $(\mathfrak{D})$-integrable, then $\lim f_n(x)$ is $(\mathfrak{D})$-integrable and $\lim (\mathfrak{D})\int_I f_n(x)dx = (\mathfrak{D})\int_I \big(\lim f_n(x)\big)dx$.

2.2. If we take $\mathfrak{GAC}$ as $\mathfrak{GAC}$ (or $\mathfrak{GAC}^*$) and $\mathfrak{AD}$ as AD (or D (ordinary derivative, that is, when $f(x)$ is considered in a set E, $Df(x)$ is the derivative concerning E), then $(\mathfrak{D})$-integral becomes general (or special) Denjoy integral. Moreover we can take $\mathfrak{GAC}$ as GAC[1] (or class of almost everywhere differentiable functions in GAC') and $\mathfrak{AD}$ as AD (or D). Thus we get two integrals, one of which is due to Denjoy and Khintchine, and the other contains an integral due to Burkill[2] as a special case. We can take also $\mathfrak{AD}$ as τ-approximate

1) In this case it requires evident modification of 4° in (2.1).

2) Burkill, Math. Zeitschrift, **34**.

derivative. Then we get the integral due to author[1]. But in this case (2.6), 5°. and then (2.7) do not hold.

§3. (3.1) R is a regular topological space satisfying the second countability axiom and having Lebesgue measure. Further in R there is a system $\mathfrak{A}$ of "intervals," that is,

1°. $\mathfrak{A}$ is a complete system of neighbourhoods.

2°. $I\varepsilon\mathfrak{A}$ implies the compactness of $\bar{I}$.

3°. For any open set 0, $(I_i;\ i=1,2,\dots,n)$ is called the decomposition of 0 if $I_i\varepsilon\mathfrak{A}$, $I_i\frown I_j=\theta$ $(i\neq j)$ and mean $(0-\bigvee_{i=1}^{n}I_i)=0$. Each I_i is called the term of decomposition. If $I'\subset I$, $I'\varepsilon\mathfrak{A}$ and $I\varepsilon\mathfrak{A}$, then there is a decomposition of I with term I'. And if $I_1\varepsilon\mathfrak{A}$, $I_2\varepsilon\mathfrak{A}$, then $I_1\frown I_2$ has a decomposition.

(3.2) If $f(I)$ is a real valued function of intervals in $I_0\subset R$, then $f(I)$ is called absolutely continuous in $E\subset I_0$ and we write $f\varepsilon\mathfrak{AC}_E$, provided that

1°. $E_1\subset E$ and $f\varepsilon\mathfrak{AC}_E$ imply $f\varepsilon\mathfrak{AC}_{E_1}$,

2°. if $E\subset I$, $f(I)$ is continuous in I and $f\varepsilon\mathfrak{AC}_E$, then $f\varepsilon\mathfrak{AC}_{\bar{E}}$, and a point function $\mathfrak{AD}f(x)$ (abstract derivative) is defined and

3°. for a closed set F, $f\varepsilon\mathfrak{AC}_F$ implies the almost everywhere existence of $\mathfrak{AD}f(x)$,

4°. if $f(I)$ is continuous in I_0 and is locally constant in the complement of a closed set F in I_0, and further if $f\varepsilon AC_F$ and $\mathfrak{AD}f(x)=0$ almost everywhere in F, then $f(I)$ is constant in I_0.

(3.3) $f(x)$ is called generalized absolutely continuous in I_0 and we writer $f\varepsilon\mathfrak{GAC}_{I_0}$, if there is a sequence of sets (E_i) such that $\bar{E}_i$ $(i=1,2,3,\dots)$ is compact, $I_0=\bigvee_{i=1}^{\infty}E_i$ and E_i is contained in a boundary set of an interval or $f\varepsilon\mathfrak{AC}_{E_1}$.

Thus we can define the integral by (2.5) which contains the Romanowski-Denjoy integral as a special case. If we introduce (2.6) 5° and 6°, then we get theorems as in §2.

3.1. We will now consider the functions with value in an abstract space X and of a real variable. Then we can define the $(\mathfrak{D})$-integral by (2.1)–(2.5).

For example, we take X as a Banach space. As $\mathfrak{AD}$ taking strong derivative, weak derivative, approximate weak derivative, pseudo-derivative and approximate pseudo-derivative, we can associate "generalized absolute continuity" to each derivative such that the condition in (2.1′) holds. Thus we can define the corresponding integrals, which contain known integrals as special case.

For example as $\mathfrak{GAC}'$ we take the class of absolutely continuous (in the Pettis sense) and almost everywhere pseudo-differentiable functions and as $\mathfrak{AD}$, pseudo-derivative. Then $(\mathfrak{D})$-integral becomes the Pettis in integral. We can get many generalizations of the Pettis integral and their Denjoy generalizations. Concerning general Dunford integral we can also get such generalization. We can similarly define

1) Izumi, this Proc. **12** (1936).

the Bochner integral as tne inverse of the strong derivative, and its Denjoy generalization. Finally we can get the integral as the inverse of weak derivative and its Denjoy generalization, which seem to be new.

For a locally convex linear topological space X we develop the above theory.

3.2. We are easy to define ($\mathfrak{D}$)-integral of (B)-space valued functions defined in R (in (3.1)).

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