38. *On the Duality Theorem of Non-commutative Compact Groups.*

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1. The duality theorem of L. Pontrjagin[1] concerning the compact commutative group was, by T. Tannaka[2], ingeneously extended to arbitrary compact group. To this extension, S. Bochner[3] and M. Krein[4] respectively gave new proofs recently. They three all start with the proof of the positivity or the continuity of certain homomorphisms. The proof given below is a direct one and will be shorter than theirs. It may be considered as a simplification of Tannaka's original proof. Our method lies in the use of Gelfand-Silov's abstraction[5] of Weierstrass' polynomial approximation theorem.

2. Let $\mathfrak{G}$ be a compact (=bicompact) Hausdorff group and let $\mathfrak{U}$ be a complete set of mutually inequivalent, continuous, unitary, irreducible representations $U(s)=\big(u_{ij}(s)\big)$ of $\mathfrak{G}$. The completeness (Peter-Weyl-Neumann's theory of almost periodic functions) implies: 1) for any pair of distinct points $s, t\in\mathfrak{G}$, there exists an $U(s)\in\mathfrak{U}$ such that $U(s)\neq U(t)$, 2) if $U_1(s)$, $U_2(s)\in\mathfrak{U}$ then the *product* (*complex conjugate*) *representation* $U_1(s)\times U_2(s)\big(\bar{U}_1(s)\big)$ is, as a unitary representation of $\mathfrak{G}$, completely reducible to a sum of a finite number of representations $\in\mathfrak{U}$. Let $\mathfrak{R}$ be the totality of *Fourier polynomials*:

$$x(s)=\sum a_{ij}^{(\gamma)}u_{ij}^{(\gamma)}(s),$$

viz. finite linear combinations of $u_{ij}^{(\gamma)}(s)$, where $\big(u_{ij}^{(\gamma)}(s)\big)\in\mathfrak{U}$ and $a_{ij}^{(\gamma)}$ denote complex numbers. By 2) $\mathfrak{R}$ is a ring with unit e $\big(e(s)\equiv 1$ on $\mathfrak{G}\big)$ and complex multipliers. Here the sum and the multiplication in $\mathfrak{R}$ is the ordinary function sum and function multiplication. Let $\mathfrak{T}$ be the totality of the linear homomorphisms T of $\mathfrak{R}$ onto the field $\mathfrak{K}$ of complex numbers such that

$$(1)\quad \begin{cases} T\cdot e=1, \\ T\cdot\bar{x}=\overline{T\cdot x} \quad \big(\text{bar indicates complex conjugates: } \bar{x}(s)=\overline{x(s)}\big). \end{cases}$$

$\mathfrak{T}$ is not void since each $s\in\mathfrak{G}$ induces such a homomorphism T_s:

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1) Topological groups, Princeton (1939).

2) Über den Dualitätssatz der nichtkommutativen topologischen Gruppen, Tôhoku Math. J., **45** (1938).

3) 位相數學, 第 **4** 卷, 第 **1** 號 (昭和 17 年).

4) On positive functionals on almost periodic functions, C. R. URSS, **30** (1941).

5) Über verschiedene Methoden der Einführung der Topologie in die Menge der maximalen Ideale eines normierten Ringes, Rec. Math., **9, 7** (1941). Cf. H. Nakano: 連續函數 ring 及ビ vector lattice, 全國紙上數學談話會 **218** (1941).

(2) $$T_s \cdot x = x(s) .$$

We have, by 1), $T_s \neq T_t$ if $s \neq t$.

$\mathfrak{T}$ may be considered as a group which contains $\mathfrak{G}$ as a subgroup.

Proof: We define the product $T=T_1T_2$ in $\mathfrak{T}$ as follows. Let $U(s)=\left(u_{ij}(s)\right)_1^n$ be any member of $\mathfrak{U}$, then we put

(3) $$T \cdot u_{ij} = \sum_{k=1}^{n} (T_1 \cdot u_{ik})(T_2 \cdot u_{kj}) \qquad (i, j=1, 2, \ldots, n).$$

Because of the linear independence of $u_{ij}^{(\gamma)}(s)$'s, $\left(u_{ij}^{(\gamma)}(s)\right) \in \mathfrak{U}$, we may extend T linearly on the whole $\mathfrak{R}$. It is easy to see that the extension T is also a member of $\mathfrak{T}$ and that $\mathfrak{T}$ is a group with the unit T_{s_0} (s_0 = the identity of $\mathfrak{G}$). $\mathfrak{G}$ is thus isomorphically embedded in the group $\mathfrak{T}$ by the correspondence $s \leftrightarrow T_s$.

Next introduce a *weak topology* in $\mathfrak{T}$ by taking for a neighbourhood of the unity T_{s_0} every set

$$\mathop{\mathcal{E}}_{T}\{|T \cdot x_i - T_{s_0} \cdot x_i| < \varepsilon, \quad x_i \in R, \quad i=1, 2, \ldots, n\}$$

$\mathfrak{T}$ is a compact Hausdorff space as a closed subset of the infinite dimensional torus. It is easily seen that the isomorphic embedding $s \leftrightarrow T_s$ is also a topological one. Hence $\mathfrak{G}$ may be considered as a closed subgroup of the compact Hausdorff group $\mathfrak{T}$. In the truth we have the

Theorem (*of T. Tannaka*). $\mathfrak{T}=\mathfrak{G}$, *viz. every* $T \in \mathfrak{T}$ *is equal to a certain* T_s:

$$T \cdot x = x(s), \qquad x \in \mathfrak{R}.$$

Proof. By the weak topology each $x(s) \in \mathfrak{R}$ may be considered as a continuous function $x(T)$ on the compact Hausdorff space $\mathfrak{T}$ such that $x(T_s)=x(s)$. The ring $\mathfrak{R}(\mathfrak{T})$ of continuous functions $x(T)$, $x \in \mathfrak{R}$, satisfies the following three conditions: i) $1=e(T) \in \mathfrak{R}(\mathfrak{T})$, ii) for any two distinct points $T_1 \neq T_2$ of $\mathfrak{T}$ there exists $x \in \mathfrak{R}$ such that $x(T_1) \neq (T_2)$, iii) for any $x(T) \in \mathfrak{R}(\mathfrak{T})$ there exists the complex conjugate function $\bar{x}(T)=\overline{x(T)}$ in $\mathfrak{R}(\mathfrak{T})$.

Now let $\mathfrak{T}-\mathfrak{G}$ be not void. Then there exists a point $T_0 \in \mathfrak{T}-\mathfrak{G}$ and a continuous function $y(T)$ on $\mathfrak{T}$ such that

(4) $$y(T) \geqq 0 \text{ on } \mathfrak{T}, \qquad y(s)=0 \text{ on } \mathfrak{G} \text{ and } y(T_0)=1.$$

By i)-iii) and the Gelfand-Silov's theorem referred to above, there exists, for any $\varepsilon > 0$, $x(s)=\sum a_{ij}^{(\gamma)} u_{ij}^{(\gamma)}(s) \in \mathfrak{R}$ such that

$$|y(T) - \sum a_{ij}^{(\gamma)} u_{ij}^{(\gamma)}(T)| \leqq \varepsilon \quad \text{on} \quad \mathfrak{T}$$

and, in particular,

$$|y(s) - \sum a_{ij}^{(\gamma)} u_{ij}^{(\gamma)}(s)| \leqq \varepsilon \quad \text{on} \quad \mathfrak{G}.$$

Let $u_{11}^{(\gamma_0)}(s)=e(s) \equiv 1$, then by taking *Haar-Neumann's mean* we obtain

$$\left|\underset{T}{M}\big(y(T)\big)-a_{11}^{(\gamma_0)}\right|\leqq\varepsilon\,,\qquad \left|\underset{s}{M}\big(y(s)\big)-a_{11}^{(\gamma_0)}\right|\leqq\varepsilon\,.$$

This is a contradiction, since we have $\underset{T}{M}\big(y(T)\big)>0$, $\underset{s}{M}\big(y(s)\big)=0$ from (4).

Q. E. D.

Remark 1. The above proof also gives a new proof of a theorem of E. R. van Kampen[1] which is an extension of W. Burnside's theorem[2]. This was kindly pointed out to me by T. Tannaka. It is to be noted that Kampen's theorem plays an important rôle in Tannaka's proof.

Remark 2. In another note the author will give a proof of Pontrjagin's duality theorem for locally compact commutative group, by combining the generalized Plancherel's theorem[3] and the theory of Haar's measure[4]. It intends to be a simplification of Raikov's proof[5], published recently.

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1) Almost periodic functions and compact groups, Ann. of Math., **37** (1936).

2) Theory of groups of finite order, 2nd ed., Cambridge (1911), 299.

3) M. Krein: Sur une généralisation du théorème de Plancherel au cas des intégrales de Fourier sur les groupes topologiques commutatifs, C, R. URSS, **30** (1941).

4) A. Weil: La réciproque du théorème de Haar dans "L'integration dans les groupes topologiques et ses applications," Paris (1940). Cf. also K. Kodaira: Über die Beziehung zwischen den Massen und Topologien in einer Gruppe, Proc. Phys.-Math. Soc. Japan, **23** (1941).

5) Generalized duality theorem for commutative groups with an invariant measure, C. R. URSS, **30** (1941).