74. On Cardinal Numbers Related with a Compact Abelian Group.

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§1. Throughout the present paper we use the following notation:

(1) $\mathfrak{p}(A)$ = the cardinal number of a set A.

Let G be a compact abelian group containing an infinite number of elements, and let us put

(2) $\mathfrak{v}(G)$ = the smallest cardinal number $\mathfrak{p}(\Gamma)$ of a system $\mathfrak{V}(0) = \{V_\gamma(0) \mid \gamma \in \Gamma\}$ of open neighborhoods $V_\gamma(0)$ of the zero element 0 of G which defines[1] the topology of G at 0,

(3) $\mathfrak{o}(G)$ = the smallest cardinal number $\mathfrak{p}(\Gamma)$ of a system $\mathfrak{O} = \{O_\gamma \mid \gamma \in \Gamma\}$ of open subsets O_γ of G which defines[2] the topology of G,

(4) $\mathfrak{d}(G)$ = the smallest cardinal number $\mathfrak{p}(D)$ of a subset D of G which is everywhere dense in G.

The purpose of the present paper is to evaluate the cardinal numbers $\mathfrak{p}(G)$, $\mathfrak{v}(G)$, $\mathfrak{o}(G)$ and $\mathfrak{d}(G)$ in terms of the cardinal number $\mathfrak{m} = \mathfrak{p}(G^*)$ of the discrete character group G^* of G. The main results may be stated as follows:

Theorem 1. $\mathfrak{p}(G) = 2^{\mathfrak{m}}$.

Theorem 2. $\mathfrak{v}(G) = \mathfrak{o}(G) = \mathfrak{m}$.

Theorem 3. $\mathfrak{d}(G) = \mathfrak{n}$, *where* $\mathfrak{n}$ *is the smallest cardinal number which satisfies* $2^{\mathfrak{n}} \geqq \mathfrak{m}$.

Theorem 1 is a generalization of the fact that a compact abelian group containing an infinite number of elements has always a cardinal number $\geqq \mathfrak{c}$, and that there is no compact abelian group whose cardinal number is exactly $\aleph_0$. Further, assuming the generalized continuum hypothesis: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, it follows from Theorem 1 that there is no compact abelian group whose cardinal number is exactly $\aleph_\alpha$ if α is a limit ordinal. Theorem 2 implies as a special case that a compact abelian group G is separable[3] (and hence metrisable) if and only if the discrete character group G^* of G is countable, and if and only if

1) A system $\mathfrak{V}(a) = \{V_\gamma(a) \mid \gamma \in \Gamma\}$ of neighborhoods $V_\gamma(a)$ of a point a of a topological space Ω defines the topology of Ω at a if, for any neighborhood $V(a)$ of a in Ω, there exists a $\gamma \in \Gamma$ such that $V_\gamma(a) \leqq V(a)$.

2) A system $\mathfrak{O} = \{O_\gamma \mid \gamma \in \Gamma\}$ of open subsets O_γ of a topological space Ω defines the topology of Ω if, for any $a \in \Omega$ and for any neighborhood $V(a)$ of a in Ω, there exists a $\gamma \in \Gamma$ such that $a \in O_\gamma \leqq V(a)$.

3) A topological space Ω is separable (=satisfies the second countability axiom of Hausdorff) if there exists a countable family $\mathfrak{O} = \{O_n \mid n = 1, 2, \dots\}$ of open subsets O_n of Ω which defines the topology of Ω.

G satisfies the first countability axiom of Hausdorff at the zero element 0 of G[4]. Finally, from Theorem 3 we see that there usually exists, in a compact abelian group G, a dense subset D of G whose cardinal number $\mathfrak{p}(D)$ is smaller than the cardinal number $\mathfrak{p}(G^*)$ of the discrete character group G^* of G, which as we already know by Theorem 2 is equal to $\mathfrak{o}(G)$. For example, in a compact abelian group G with $\mathfrak{p}(G)=2^{\mathfrak{c}}$ $\big($i.e. with $\mathfrak{p}(G^*)=\mathfrak{c}$ because of Theorem 1$\big)$, there always exists a countable subset D of G (or even a countable subgroup H of G) which is dense in G. This fact, however, is not surprising since we already know[5] that there exists a monothetic or a solenoidal compact abelian group which is not separable. Theorem 3 only shows that this is quite a natural phenomenon. If we again assume the generalized continuum hypothesis, then $\mathfrak{n}=\mathfrak{m}$ if and only if $\mathfrak{m}=\aleph_\alpha$ with a limit ordinal α, and $\mathfrak{n}=\aleph_\alpha$ if $\mathfrak{m}=\aleph_{\alpha+1}$.

Theorem 1, 2 and 3 are all clear if $\mathfrak{m}=\aleph_0$. Hence, throughout the rest of this paper we always assume that $\mathfrak{m}>\aleph_0$.

§ 2. *Proof of Theorem 1.* Let G be a compact abelian group containing an infinite number of elements, and let G^* be the discrete character group of G. Since every $a \in G$ can be considered as a real-valued (mod. 1) function[6] $\chi(a^*)=(a, a^*)$ defined on G^*, and since for any pair $\{a, b\} \subseteqq G$ with $a \neq b$ there exists an $a^* \in G^*$ with $(a, a^*) \neq (b, a^*)$, so we see that $\mathfrak{p}(G) \leqq \mathfrak{c}^{\mathfrak{m}}=2^{\mathfrak{m}}$.

In order to show that $\mathfrak{p}(G) \geqq 2^{\mathfrak{m}}$, let us observe how a character $\chi(a^*)$ on G^* can be defined constructively by transfinite induction: Let

$$(5) \qquad G^*=\{a_\alpha^* \mid 0 \leqq \alpha < \omega(\mathfrak{m})\},$$

be a well-ordering of all elements of G^* such that $a_0^*=0^*$ (= the zero element of G^*), where $\omega(\mathfrak{m})$ is the smallest ordinal number which corresponds to the cardinal number $\mathfrak{m}$. Let us divide G^* into three classes A_1^*, A_2^* and A_3^*: the first class A_1^* consists of $a_0^*=0^*$ and of all a_α^* which is contained in a subgroup H_α^* of G^* generated by $\{a_\beta^* \mid 0 \leqq \beta < \alpha\}$; the second class A_2^* consists of all a_α^* such that $a_\alpha^* \bar{\in} H_\alpha^*$ and $m a_\alpha^* \in H_\alpha^*$ for some integer $m>1$; and finally the third class A_3^* consists of all a_α^* such that $m a_\alpha^* \bar{\in} H_\alpha^*$ for $m=1, 2, \ldots$. It is then easy to see that A_2^* and A_3^* together generate G^*, and so $\mathfrak{p}(A_2^* \cup A_3^*)=\mathfrak{m}$, since by assumption $\mathfrak{m}>\aleph_0$. Let us now define a character $\chi(a^*)$ on G^* constructively by transfinite induction: for each $a_\alpha^* \in A_1^*$, the value $\chi(a_\alpha^*)$ is uniquely determined by the values $\{\chi(a_\beta^*) \mid \beta<\alpha\}$; for each $a_\alpha^* \in A_2^*$, let m_α be the smallest positive integer such that $m_\alpha a_\alpha^* \in H_\alpha^*$ Then there are exactly m_α different possibilities to define $\chi(a_\alpha^*)$, namely,

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4) S. Kakutani, Über die Metrisation der topologischen Gruppen, Proc. **12** (1936), 82–84.

5) H. Anzai and S. Kakutani, Bohr compactifications of a locally compact abelian group, to appear in Proc. **19** (1943).

6) (a, a^*) denotes the value of a character $a^* \in G^*$ at a point $a \in G$, and also the value of a character $a \in G$ at a point $a^* \in G^*$.

(6) $\chi(a_a^*)=\frac{1}{m_a}\sum_{p=1}^k n_p a_{\beta_p}^* + \frac{j}{m_a}$ (mod. 1), $j=0, 1, \dots, m_a-1$

if

(7) $m_a a_a^* = \sum_{p=1}^k n_p a_{\beta_p}^* \in H_a^*$, $0<\beta_1<\cdots<\beta_n<a$.

Finally, for each $a_a^* \in A_3^*$, the value $\chi(a_a^*)$ can be chosen arbitrarily (mod. 1). From these facts follows immediately that $\mathfrak{p}(G) \geqq 2^{\mathfrak{p}(A_2^* \smile A_3^*)} = 2^{\mathfrak{m}}$, as we wanted to prove. This completes the proof of Theorem 1.

§ 3. *Proof of Theorem 2.* Let G^* be the discrete character group of a compact abelian group G. It is easy to see that a defining neighborhood system $\mathfrak{V}(0)=\{V_\gamma(0) \mid \gamma \in \Gamma\}$ of the zero element 0 of G is given by

(8) $$V_\gamma(0)=\left\{a \mid |(a, a_p^*)| < \frac{1}{m},\ p=1, \dots, k\right\}$$

(9) $$\Gamma=\left\{\gamma=\{a_1^*, \dots, a_k^* ; m\} \mid \{a_1^*, \dots, a_k^*\} \subseteqq G^* ; k, m=1, 2, \dots\right\}.$$

From this follows easily that $\mathfrak{v}(G) \leqq \mathfrak{p}(\Gamma)=\mathfrak{p}(G^*)=\mathfrak{m}$.

In order to show that $\mathfrak{v}(G) \geqq \mathfrak{m}$, let $\mathfrak{V}(0)=\{V_\gamma(0) \mid \gamma \in \Gamma\}$ be a family of neighborhoods $V_\gamma(0)$ of the zero element 0 of G which defines the topology of G at 0 and such that $\mathfrak{p}(\Gamma)=\mathfrak{v}(G)$. For each $\gamma \in \Gamma$, let H_γ be a closed subgroup of G contained in $V_\gamma(0)$ such that the factor group $F_\gamma=G/H_\gamma$ is a compact separable abelian group. It is then clear that the discrete character group F_γ^* of F_γ is countable. Let us consider F_γ^* as the family of all continuous characters on G which vanish identically on H_γ. F_γ^* is then a subgroup of G^*, and we claim that

(10) $$G^* = \bigvee_{\gamma \in \Gamma} F_\gamma^*.$$

In order to prove (10), let a_0^* be an arbitrary element of G^* and let us put

(11) $$V_0(0)=\left\{a \mid |(a, a_0^*)| < \frac{1}{4}\right\}$$

Then $V_0(0)$ is an open neighborhood of the zero element 0 of G. Let now $\gamma \in \Gamma$ be such that $V_\gamma(0) \subseteqq V_0(0)$, and let H_γ be a closed subgroup of G contained in $V_\gamma(0)$ as defined above. Then $a \in H_\gamma$ implies $na \in H_\gamma$, hence $|(na, a_0^*)| < 1/4$ (mod. 1) for $n=1, 2, \dots$ and consequently $(a, a_0^*) = 0$. Thus the character $\chi(a)=(a, a_0^*)$ vanishes identically on H_γ, and so we must have $a_0^* \in F_\gamma^*$. Since a_0^* is an arbitrary element of G^*, this proves (10). From (10) follows immediately that $\mathfrak{m}=\mathfrak{p}(G^*) \leqq \mathfrak{p}(\Gamma) = \mathfrak{v}(G)$.

We shall next show that $\mathfrak{o}(G)=\mathfrak{v}(G)$. It is clear that $\mathfrak{o}(G) \geqq \mathfrak{v}(G)$. In order to prove that $\mathfrak{o}(G) \leqq \mathfrak{v}(G)$, let $\mathfrak{V}(0)=\{V_\gamma(0) \mid \gamma \in \Gamma\}$ be a family of open neighborhoods $V_\gamma(0)$ of the zero element 0 of G which defines the topology of G at 0. For each $\gamma \in \Gamma$, take a covering $G \subseteqq \bigvee_{i=1}^{n_\gamma} O_{\gamma,i}$ of G by a finite number of translations $O_{\gamma,i}=a_{\gamma,i}+V_\gamma(0)$ of $V_\gamma(0)$. Then we claim that $\mathfrak{O}=\{O_{\gamma,i} \mid i=1, \dots, n_\gamma ; \gamma \in \Gamma\}$ is a family of open subsets of G which defines the topology of G.

In fact, for any $a \in G$ and for any open set $O(a)$ containing a, let $\beta \in \Gamma$ be such that $a+V_\beta(0) \leqq O(a)$. Then take a $\gamma \in \Gamma$ such that $V_\gamma(0)-V_\gamma(0) \leqq V_\beta(0)$ and also a translation $O_{\gamma,i}=a_{\gamma,i}+V_\gamma(0)$ of $V_\gamma(0)$ which contains a. Then we see $a \in a_{\gamma,i}+V_\gamma(0)=a+V_\gamma(0)-(a-a_{\gamma,i}) \leqq a+V_\gamma(0)-V_\gamma(0) \leqq a_0+V_\beta(0) \leqq O(a)$. Thus $\mathfrak{O}=\{O_{\gamma,i} \mid i=1,\dots,n_\gamma;\ \gamma \in \Gamma\}$ defines the topology of G. From this follows immediately that $\mathfrak{o}(G) \leqq \mathfrak{p}(\mathfrak{O})=\mathfrak{p}(\Gamma)=\mathfrak{m}$. This completes the proof of Theorem 2.

§ 4. *Proof of Theorem 3.* Let G be a compact abelian group with $\mathfrak{p}(G)=2^{\mathfrak{m}}$, or what amounts to the same thing by Theorem 1, with $\mathfrak{p}(G^*)=\mathfrak{m}$, where we denote as usual by G^* the discrete character group of G.

Let D be a subset of G which is dense in G with $\mathfrak{p}(D)=\mathfrak{n}$. We shall show that $\mathfrak{m} \leqq 2^{\mathfrak{n}}$. In order to show this, let H be a subgroup of G which is generated by D. Since D is obviously an infinite set, so we see $\mathfrak{p}(D)=\mathfrak{p}(H)=\mathfrak{n}$. Let us now consider H as a discrete group, and let H^* be the compact character group of H. Then every continuous character $\chi(a)=(a, a^*)$ on G may be considered as an algebraic character on H, and so there exists an algebraic homomorphism $a^{*\prime}=\varphi^*(a^*)$ of G^* onto an algebraic subgroup $G^{*\prime}$ of H^*[7]. This homomorphism is even an isomorphism since H is dense in G. Thus G^* is algebraically isomorphic with an algebraic subgroup $G^{*\prime}$ of H^* and hence $\mathfrak{m}=\mathfrak{p}(G^*)=\mathfrak{p}(G^{*\prime}) \leqq \mathfrak{p}(H^*)=2^{\mathfrak{n}}$ by Theorem 1. This completes the first half of the proof of Theorem 3[8].

Let now $\mathfrak{n}$ be a cardinal number satisfying $\mathfrak{m} \leqq 2^{\mathfrak{n}}$. We shall show that there exists a subset D of G with $\mathfrak{p}(D) \leqq \mathfrak{n}$ which is dense in G. For this purpose it suffices to prove the following

Theorem 4. Let G^* be a discrete abelian group with $\mathfrak{p}(G^*)=\mathfrak{m}$, and let $\mathfrak{n}$ be a cardinal number which satisfies $\mathfrak{m} \leqq 2^{\mathfrak{n}}$. Then there exists a family $D=\{\chi(a^*)\}$ of algebraic characters on G^* with $\mathfrak{p}(D) \leqq \mathfrak{n}$ which separates every element $a^* \in G^*$ with $a^* \neq 0^*$ from 0^* $\Big($i.e. such that, for any $a^* \in G^*$ with $a^* \neq 0^*$, there exists a character $\chi \in D$ with $\chi(a^*) \neq 0\Big)$.

In fact, if there exists such a family D, then D may be considered as a subset of the compact character group $G=G^{**}$ of G^*. The algebraic subgroup H of G which is generated by D is dense in G; for, otherwise, there would exist an element $a^* \in G^*$ such that $(a, a^*)=0$ for any $a \in H$, or equivalently $\chi(a^*)=0$ for any $\chi \in D$, in contradiction with the separating property of $D=\{\chi(a^*)\}$ stated above.

So it only remains to prove Theorem 4.

Proof of Theorem 4. We shall divide our arguments into three cases:

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7) H. Anzai and S. Kakutani, loc. cit. 5).

8) We may obtain the same inequality $\mathfrak{m} \leqq 2^{\mathfrak{n}}$ directly by appealing to the fact that if a Hausdorff space Ω contains a dense subset D with $\mathfrak{p}(D)=\mathfrak{n}$, the cardinal number $\mathfrak{p}(\Omega)$ of the space Ω must satisfy $\mathfrak{p}(\Omega) \leqq 2^{2^{\mathfrak{n}}}$ (Cf. B. Pospisil, Annals of Math. **38** (1937)). But in order to obtain $\mathfrak{m} \leqq 2^{\mathfrak{n}}$ from $2^{\mathfrak{m}} \leqq 2^{2^{\mathfrak{n}}}$ we need the generalized continuum hypothesis.

1st case: G^* has no element of finite order. We shall first notice that there exists a subset $B^*=\{b_\gamma^* \mid \gamma \in \Gamma\}$ of G^* with $\mathfrak{p}(B^*)=\mathfrak{p}(\Gamma)=\mathfrak{m}=\mathfrak{p}(G^*)$ consisting of mutually independent elements and such that every $a^* \in G^*$ with $a^* \neq 0^*$ satisfies a relation of the form:

$$ma^* = \sum_{p=1}^{k} n_p b_{\gamma_p}, \tag{12}$$

where $\{\gamma_1, \dots, \gamma_k\} \subseteqq \Gamma$ and $\{m, n_1, \dots, n_k\}$ is a finite system of positive or negative integers.

In fact, it suffices to take as B^* any maximal subset of G^* consisting of mutually independent elements, whose existence is clear from Zorn's lemma. It is then clear that every $a^* \in G^*$ with $a^* \neq 0^*$ satisfies a relation of the form (12). Further, since G^* has no element of finite order, for any given finite systems $\{\gamma_1, \dots, \gamma_k\} \subseteqq \Gamma$ and $\{m, n_1, \dots, n_k\}$, there exists at most one element $a^* \in G^*$ which satisfies (12). From this follows immediately that $\mathfrak{p}(G^*)=\mathfrak{p}(B^*)$ if we remember that $\mathfrak{m} > \aleph_0$ by assumption.

Let H^* be an algebraic subgroup of G^* generated by B^*. Then for any system $\{c_\gamma \mid \gamma \in \Gamma\}$ of real numbers (mod. 1), there exists a uniquely determined algebraic character $\chi(a^*)$ defined on H^* which satisfies $\chi(b_\gamma^*)=c_\gamma$ (mod. 1) for any $\gamma \in \Gamma$.

Let now $\mathfrak{D}=\{\Delta_\sigma \mid \sigma \in \sum\}$ be a family of diadic partitions Δ_σ of Γ: $\Gamma=\Gamma_\sigma \cup \Gamma'_\sigma$, $\Gamma_\sigma \cap \Gamma'_\sigma=\theta$, with $\mathfrak{p}(\sum) \leqq \mathfrak{n}$ satisfying the following condition[9]: for any finite system $\{\gamma_1, \dots, \gamma_k\} \subseteqq \Gamma$ with $\gamma_i \neq \gamma_j$, for $i \neq j$, there exists a $\sigma \in \sum$ such that $\gamma_1 \in \Gamma_\sigma$ and $\{\gamma_2, \dots, \gamma_k\} \subseteqq \Gamma'_\sigma$. The existence of such a family $\mathfrak{D}$ is an easy consequence of the fact that $\mathfrak{p}(\Gamma)=\mathfrak{m}$ and $\mathfrak{m} \leqq 2^{\mathfrak{n}}$. In fact, it is easy to see that there exists a family $\mathfrak{D}_0=\{\Delta_\tau^0 \mid \tau \in T\}$ of diadic partitions Δ_τ^0 of Γ: $\Gamma=\Gamma_\tau^0 \cup \Gamma_\tau^{0\prime}$, $\Gamma_\tau^0 \cap \Gamma_\tau^{0\prime}=\theta$ with $\mathfrak{p}(T) \leqq \mathfrak{n}$ satisfying the condition that for any pair $\{\gamma_1, \gamma_2\} \subseteqq \Gamma$ with $\gamma_1 \neq \gamma_2$, there exists a $\tau \in T$ such that $\gamma_1 \in \Gamma_\tau^0$ and $\gamma_2 \in \Gamma_\tau^{0\prime}$. It is then clear that the family $\mathfrak{D}=\{\Delta_\sigma \mid \sigma \in \sum\}$ of all diadic partitions Δ_σ of Γ: $\Gamma=\Gamma_\sigma \cup \Gamma'_\sigma$ where $\Gamma_\sigma=\Gamma_{\tau_1}^0 \cap \cdots \cap \Gamma_{\tau_n}^0$, $\Gamma'_\sigma=\Gamma_{\tau_1}^{0\prime} \cup \cdots \cup \Gamma_{\tau_n}^{0\prime}$, $\sum=\left\{\sigma=\{\tau_1, \dots, \tau_n\} \mid \{\tau_1, \dots, \tau_n\} \subseteqq T;\ n=1,2,\dots\right\}$ is a required one.

Now, for any $\sigma \in \sum$, let us define a character $\chi_\sigma(a^*)$ on H^* by giving the values $\{\chi_\sigma(b_\gamma^*) \mid \gamma \in \Gamma\}$ as follows: $\chi_\sigma(b_\gamma^*)=\lambda_0$ if $\gamma \in \Gamma_\sigma$ and $\chi_\sigma(b_\gamma^*)=0$ if $\gamma \in \Gamma'_\sigma$, where λ_0 is a fixed irrational number independent of σ and γ. This character $\chi_\sigma(a^*)$ can then be extended to a character $\bar{\chi}_\sigma(a^*)$ on G^*. The extension is not unique unless $H^*=G^*$; so take any of the possible extensions. We claim that $D=\{\bar{\chi}_\sigma(a^*) \mid \sigma \in \sum\}$ is a required family, i.e. that for any $a^* \in G^*$ with $a^* \neq 0^*$, there exists a $\sigma \in \sum$ such that $\bar{\chi}_\sigma(a^*) \neq 0$. In fact, every $a^* \in G^*$ with $a^* \neq 0^*$ satisfies a relation of the form (12). Let $\sigma \in \sum$ be such that $\gamma_1 \in \Gamma_\sigma$ and $\{\gamma_2, \dots, \gamma_k\} \subseteqq \Gamma'_\sigma$. Then $\bar{\chi}_\sigma(ma^*)=\chi_\sigma(ma^*)=\chi_\sigma(\sum_{p=1}^{k} n_p b_{\gamma_p}^*)=n_1\lambda_0 \not\equiv 0$ (mod. 1), and so $\bar{\chi}_\sigma(a^*) \not\equiv 0$ (mod. 1). This completes the proof of Theorem 4 in case G^* has no element of finite order.

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9) In case $k=1$, this condition only means that $\gamma_1 \in \Gamma_\sigma$.

2nd case: every element of G^* is of finite order. Let G_n^* be a subgroup of G^* consisting of all elements $a^* \in G^*$ which satisfy $na^* = 0^*$ We have clearly $G^* = \bigcup_{n=1}^{\infty} G_n^*$, and $\mathfrak{p}(G_n^*) \leqq 2^{\mathfrak{n}}$, $n=1,2,\dots$. By a result of G. Köthe[10], each G_n^* is algebraically isomorphic with a restricted infinite direct sum of a family $\{C_\gamma \mid \gamma \in \Gamma_n\}$ of finite cyclic groups C_γ whise degree d_γ divides n:

$$G_n^* = \sum_{\gamma \in \Gamma_n} \oplus C_\gamma. \tag{13}$$

Consider each C_γ as a subgroup of G_n^*, and let b_γ^* be a generating element of C_γ. Then, (13) means that every element $a^* \in G_n^*$ with $a^* \neq 0^*$ may be expressed in the form:

$$a^* = \sum_{p=1}^{k} n_p b_{\gamma_p}^*, \tag{14}$$

where $\{\gamma_1, \dots, \gamma_k\} \subseteqq \Gamma_n$ and $\{n_1, \dots, n_k\}$ is a finite system of positive integers such that $0 < n_p < d_{\gamma_p}$ for $p=1, \dots, k$. It is clear that $\mathfrak{p}(\Gamma_n) \leqq \mathfrak{m}$. Since the compact character group $(G_n^*)^*$ of G_n^* is topologically isomorphic with the unrestricted infinite direct sum of the same family $\{C_\gamma \mid \gamma \in \Gamma_n\}$ of cyclic groups:

$$(G_n^*)^* = \sum_{\gamma \in \Gamma_n} \oplus C_\gamma, \tag{15}$$

so we see that for any system $\{c_\gamma \mid \gamma \in \Gamma_n\}$ of real numbers $c_\gamma = n_\gamma^* / d_\gamma$ where n_γ^* is an integer satisfying $0 \leqq n_\gamma^* < d_\gamma$, there exists a uniquely determined character $\chi(a^*)$ on G_n^* such that $\chi(b_\gamma^*) = c_\gamma = n_\gamma^*/d_\gamma$ for any $\gamma \in \Gamma_n$, and so

$$\chi(a^*) = \sum_{p=1}^{k} \frac{n_p n_{\gamma_p}^*}{d_{\gamma_p}} \tag{16}$$

if a^* is of the form (14).

Let us again take a family $\mathfrak{D} = \{\Delta_\sigma \mid \sigma \in \sum_n\}$ of diadic partitions Δ_σ of Γ_n: $\Gamma_n = \Gamma_\sigma \cup \Gamma'_\sigma$, $\Gamma_\sigma \cap \Gamma'_\sigma = \theta$, with $\mathfrak{p}(\sum_n) = \mathfrak{n}$ satisfying the same conditions as in above. Then, for each $\sigma \in \sum_n$, let us define a character $\chi_\sigma(a^*)$ on G_n^* by giving the values $\{\chi_\sigma(b_\gamma^*) \mid \gamma \in \Gamma_n\}$ as follows: $\chi_\sigma(b_\gamma^*) = 1/d_\gamma$ if $\gamma \in \Gamma_\sigma$ and $\chi_\sigma(b_\gamma^*) = 0$ if $\gamma \in \Gamma'_\sigma$. It is then easy to see that the family $D_n = \{\chi_\sigma(a^*) \mid \sigma \in \sum_n\}$ of characters thus obtained has a required separating property for G_n^*. In fact, every $a^* \in G_n^*$ with $a^* \neq 0^*$ may be expressed in the form (14), and if we take a $\sigma \in \sum_n$ such that $\gamma_1 \in \Gamma_0$ and $\{\gamma_2, \dots, \gamma_k\} \subseteqq \Gamma'_\sigma$, then it is clear that $\chi_\sigma(a^*) = n_1/d_{\gamma_1} \not\equiv 0$ (mod. 1).

Thus, for each n, we have obtained a family $D_n = \{\chi_\sigma(a^*) \mid \sigma \in \sum_n\}$ of characters on G_n^* having a requireo separating property for G_n^*. Extend each $\chi_\sigma(a^*) \in D_n$ to a character $\bar{\chi}_\sigma(a^*)$ on G^*. This extension is not unique unless $G^* = G_n^*$; so take any of the possible extensions. If we denote by $\bar{D}_n$ the family $\{\bar{\chi}_\sigma(a^*) \mid \sigma \in \sum_n\}$ of characters thus obtained by extension, then it is clear that $D = \bigcup_{n=1}^{\infty} \bar{D}_n$ is a required family for G^*. Thus Theorem 4 is proved in case every element of G^* is of finite order.

10) G. Köthe, Mathematische Annalen, **105** (1931), 15-39.

3rd case: case of a general discrete abelian group G^*. Let G_0^* be a subgroup of G^* consisting of all elements of G^* of finite order. Then the factor group $F^*=G^*/G_0^*$ has no element of finite order. It is clear that $\mathfrak{p}(G_0^*)\leqq\mathfrak{p}(G^*)\leqq 2^{\mathfrak{n}}$ and $\mathfrak{p}(F^*)=\mathfrak{p}(G^*/G_0^*)\leqq\mathfrak{p}(G^*)\leqq 2^{\mathfrak{n}}$. Hence, by the results obtained in the first and the second cases, there exist a family D_0 of characters on G^* with $\mathfrak{p}(D_0)\leqq\mathfrak{n}$ which separates every $a^*\in G_0^*$ with $a^*\neq 0^*$ from 0^*, and a family D' of characters on $F^*=G^*/G_0^*$ which separates every element $a^{*\prime}\in F^*$ with $a^{*\prime}\neq 0^{*\prime}$ from $0^{*\prime}$, where $0^{*\prime}$ is the zero element of F^*. Extend each character $\chi(a^*)\in D_0$ to a character $\chi(a^*)$ on G^* in any possible way, and let $\bar{D}_0$ be the family of all characters thus extended. Further, consider every character $\chi'(a^{*\prime})\in D'$ on $F^*=G^*/G_0^*$ as a character $\bar{\chi}'(a^*)$ on G^* which vanishes identically on G_0^*, and let $\bar{D}'$ be the family of characters on G^* thus obtained. It is then easy to see that $D=\bar{D}_0\cup\bar{D}'$ is a family of characters on G^* with a required separating property for G^*.

This completes the proof of Theorem 4 in a general case.

Incidentally, we have proved the following

Theorem 5. Let G^* be a discrete abelian group with $\mathfrak{p}(G^*)=\mathfrak{m}$, and let $\mathfrak{n}$ be a cardinal number which satisfies $\mathfrak{m}\leqq 2^{\mathfrak{n}}$. Then there exists a compact abelian group H^* with $\mathfrak{p}(H^*)\leqq 2^{\mathfrak{n}}$ which contains an algebraic subgroup algebraically isomorphic with G^*

§ 5. *Problems.* It would be an interesting problem to investigate how far we can obtain analogous results for non-commutative compact groups. And how is the situation for locally compact groups? We may also ask the same questions for homogeneous topological spaces, where we mean under a homogeneous topological space a topological space Ω such that, for any pair of points $\{a,b\}\subseteq\Omega$ there exists a homeomorphism of Ω onto itself which maps a onto b.