73. Normed Ring of a Locally Compact Abelian Group.

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(Comm. by T. TAKAGI, M.I.A., July 12, 1943.)

§ 1. Let G be a locally compact (not necessarily separable) abelian group, and let $L^2(G)$ be the generalized Hilbert space of all complex-valued functions $x(g)$ which are defined, measurable and square integrable on G with respect to a Haar measure of G (with a certain fixed normalization) having

$$(1) \qquad \|x\|=\left(\int_G |x(g)|^2 dg\right)^{\frac{1}{2}}$$

as its norm. Let further $\mathfrak{B}(G)$ be the ring of all bounded linear transformations B which map $L^2(G)$ into itself. Then $\mathfrak{B}(G)$ is a (non-commutative) normed ring[1] with respect to the norm

$$(2) \qquad |||B||| = \sup_{|x|\leq 1} \|B(x)\| .$$

For each $a \in G$, let us denote by U_a a unitary transformation of $L^2(G)$ onto itself which is defined by

$$(3) \qquad U_a(x)=x_a , \qquad x_a(g)=x(g-a) .$$

Then $\mathfrak{U}(G)=\{U_a \mid a \in G\}$ is a group of unitary transformations which is algebraically isomorphic with G. Let further $\mathfrak{A}(G)$ be an algebraic subring of $\mathfrak{B}(G)$ which is generated by $\mathfrak{U}(G)$, i. e. a subring of $\mathfrak{B}(G)$ consisting of all $A \in \mathfrak{B}(G)$ of the form:

$$(4) \qquad A=\sum_{p=1}^{k} \alpha_p U_{a_p} ,$$

where $\{a_1, \ldots, a_k\} \subseteqq G$ and $\{\alpha_1, \ldots, \alpha_k\}$ is an arbitrary finite system of complex numbers. Let further $\mathfrak{R}(G)$ be the closure of $\mathfrak{A}(G)$ in $\mathfrak{B}(G)$, i. e. a subring of $\mathfrak{B}(G)$ consisting of all $B \in \mathfrak{B}(G)$ such that for any $\varepsilon > 0$ there exists an $A \in \mathfrak{A}(G)$ satisfying $|||B-A||| < \varepsilon$.

The purpose of this paper is to determine a general form of maximal ideals of $\mathfrak{R}(G)$. It will be shown that there exists a one-to-one correspondence between the family $\mathfrak{M}(G)$ of all maximal ideals M of $\mathfrak{R}(G)$ and the family $\mathfrak{X}(G)$ of all algebraic (=not necessarily continuous) characters[2] $\chi(a)$ defined on G. This correspondence is even

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1) I. Gelfand, Normierte Ringe, Recueil Math., **9** (1941), 3-25.

2) Under a *character* of a locally compact abelian group G, we understand a continuous representation of G by the additive group of real numbers mod. 1. Sometimes it is also necessary to consider representations of G which are not necessarily continuous. In order to distinguish these cases, we usually say *continuous characters* and *algebraic characters* of G.

a homeomorphism if we take the usual weak topology of $\mathfrak{M}(G)$ with respect to which $\mathfrak{M}(G)$ is a compact Hausdorff space, and if we consider $\mathfrak{X}(G)$ as the compact character group $G^{(d)*}$ of a discrete abelian group $G^{(d)}$ which is algebraically isomorphic with G. It will also be shown that $\mathfrak{R}(G)$ is isometrically isomorphic with the normed ring $C(\mathfrak{M}(G))=C(\mathfrak{X}(G))$ of all complex-valued continuous functions defined on $\mathfrak{M}(G)=\mathfrak{X}(G)$. From these two facts follows immediately that the normed ring $\mathfrak{R}(G)$ is uniquely determined up to an isometric isomorphism by the algebraic structure of a locally compact abelian group G, and so is independent of the topology or the Haar measure of G which we needed in defining $L^2(G)$. Thus it turns out that in order to investigate the normed ring $\mathfrak{R}(G)$ of a locally compact abelian group G, it suffices to discuss the case when G is a diserete abelian group.

§ 2. Let M be an arbitrary maximal ideal of $\mathfrak{R}(G)$. Then there exists a continuous natural homomorphism $B\to\varphi_M(B)$ of $\mathfrak{R}(G)$ onto the ring of complex numbers such that $|\varphi_M(B)|\leqq |||B|||$ for any $B\in\mathfrak{R}(G)$ and $M=\{B\,|\,\varphi_M(B)=0\}$. It is then clear that $a\to U_a\to\varphi_M(U_a)$ is an algebraic representation of G by complex numbers. Further, since $|\varphi_M(U_a)|\leqq |||U_a|||=1$ and $|(\varphi_M(U_a))^{-1}|=|\varphi_M(U_{-a})|\leqq |||U_{-a}|||=1$, so we see that $|\varphi_M(U_a)|=1$ for all $a\in G$. Thus $\varphi_M(U_a)=\exp\left(2\pi i\chi_M(a)\right)$ defines an algebraic character $\chi_M(a)$ on G (whose value is a real number mod. 1), of which we do not know whether it is continuous or not. In the following lines we shall show that $\chi_M(a)$ is not necessarily continuous unless G is discrete, and that every algebraic character $\chi(a)$ of G may be obtained in this way, i.e. that for any algebraic character $\chi(a)$ defined on G there exists a maximal ideal M of the normed ring $\mathfrak{R}(G)$ such that $\chi(a)=\chi_M(a)$ for all $a\in G$.

§ 3. Let G^* be the character group of G in the sense of L. Pontrjagin[1] and E. R. van Kampen[2] G^* is also a locally compact abelian group. Hence we may consider the generalized Hilbert space $L^2(G^*)$ and the ring $B(G^*)$ of all bounded linear transformations B^* of $L^2(G^*)$ into itself. The norm of an element $x^*\in L^*(G^*)$ is denoted by $\|x^*\|$, and the norm of a transformation $B^*\in\mathfrak{B}(G^*)$ is denoted by $|||B^*|||$. It is known that if we take a suitable normalization of a Haar measure on G^*, then an analogue of Plancherel's theorem is true[3][4]: for any $x(g)\in L^2(G)$, the integral[5]

$$x^*(g^*)=\int_G x(g)\exp\left(2\pi i(g,g^*)\right)dg\,, \tag{5}$$

1) L. Pontrjagin, Topological Groups, Princeton, 1939.

2) E. R. van Kampen, Locally bicompact abelian groups and their character groups, Annals of Math., **36** (1935), 448–463.

3) A. Weil, Intégrations dans les groupes et leurs applications, Actualités, Paris, 1940.

4) M. Krein, Sur une généralisation du théorème de Plancherel au cas des intégrales de Fourier sur les groupes topologiques commutatifs, C. R. URSS, **30** (1940), 484–488.

5) (g,g^*) denotes the value of a character $g^*\in G^*$ at a point $g\in G$, and also the value of a character $g\in G$ at a point $g^*\in G^*$

which exists in the sense of the limit in mean, defines an element $x^* = P(x) \in L^2(G^*)$; conversely, for any $x^*(g^*) \in L^2(G^*)$, the integral

$$x(g) = \int_{G^*} x^*(g^*) \exp\big(-2\pi i(g, g^*)\big) dg^* , \tag{6}$$

which again exists in the sense of the limit in mean, defines an element $x = Q(x^*) \in L^2(G)$; both P and Q are isometric linear transformations and are inverse to each other: $\|P(x)\| = \|x\|$, $\|Q(x^*)\| = \|x^*\|$ and $PQ = QP = I$ (=identity).

For any $B \in \mathfrak{B}(G)$, let us consider an element $B^* \in \mathfrak{B}(G^*)$ defined by $B^* = PBQ$. It is clear that B is obtained from B^* by the inverse relation: $B = QB^*P$, and that the correspondence $B \leftrightarrow B^*$ gives an isometric isomorphism of $\mathfrak{B}(G)$ onto $\mathfrak{B}(G^*)$. Let us now consider a subring $\mathfrak{R}^*(G^*)$ of $\mathfrak{B}(G^*)$ which corresponds to $\mathfrak{R}(G)$ by this isomorphism. First it is easy to see that if $A = U_a$, then the corresponding A^* is a bounded linear transformation of $L^2(G^*)$ which maps $x^*(g^*)$ to $\exp\big(2\pi i(a, g^*)\big)x^*(g^*)$. Further, if A is of the form (4), then the corresponding A^* is a bounded linear transformation of $L^2(G^*)$ which maps $x^*(g^*)$ to $\Big(\sum_{p=1}^k a_p \exp\big(2\pi i(a_p, g^*)\big)\Big)x^*(g^*)$. From this follows easily that

$$|||\textstyle\sum_{p=1}^k a_p U_{a_p}||| = \sup_{g^* \in G^*} \Big|\sum_{p=1}^k a_p \exp\big(2\pi i(a_p, g^*)\big)\Big| . \tag{7}$$

Thus the norm of a transformation $A = \sum_{p=1}^k a_p U_{a_p}$ coincides with the norm $\|f_A^*\|$ of a complex-valued continuous function

$$f_A^*(g^*) = \textstyle\sum_{p=1}^k a_p \exp\big(2\pi i(a_p, g^*)\big) , \tag{8}$$

where we put as usual

$$\|f^*\| = \sup_{g^* \in G^*} |f^*(g^*)| . \tag{9}$$

Let now $BAP(G^*)$ be the family of all complex-valued Bohr almost periodic[1] functions $f^*(g^*)$ defined on G^*. $BAP(G^*)$ is a normed ring with (9) as its norm, and the fact observed above shows that $\mathfrak{A}(G)$ is isometrically isomorphic with a subring $FLC(G^*)$ of $BAP(G^*)$ consisting of all finite linear combinations $\sum_{p=1}^k a_p \exp\big(2\pi i(a_p, g^*)\big)$ of exponential continuous characters $\exp\big(2\pi i(a_p, g^*)\big)$ defined on G^*. Since for any $f^*(g^*) \in BAP(G^*)$ and for any $\varepsilon > 0$, there exists an $f_A^*(g^*) \in FLC(G^*)$ such that $\|f^* - f_A^*\| < \varepsilon$, so we see that the subring $\mathfrak{R}^*(G^*)$ of $\mathfrak{B}(G^*)$ which corresponds to $\mathfrak{R}(G)$ by the isomorphism stated above consists exactly of all bounded linear transformations of

1) A complex-valued function $f^*(g^*)$ defined on a locally compact abelian group G^* is a *Bohr almost periodic function*, if $f^*(g)$ is uniformly continuous on G^* and if the family $\{f^*_{a^*}(g^*) \mid a^* \in G^*\}$ of all translations $f^*_{a^*}(g^*) = f^*(g^* + a^*)$ of $f^*(g^*)$ is totally bounded with respect to the metric defined by the norm (9). Cf. J. von Neumann, On almost periodic functions in groups, Trans. Amer. Math. Soc. **36** (1934), 446-492.

$L^2(G^*)$ which maps $x^*(g^*)$ to $f^*(g^*)x^*(g^*)$, where $f^*(g^*)$ is a complex-valued Bohr almost periodic function defined on G^*. Thus

Theorem 1. The normed ring $\mathfrak{R}(G)$ *of a locally compact abelian group* G *is isometrically isomorphic with the normed ring* $BAP(G^*)$ *of all complex-valued Bohr almost periodic functions* $f^*(g^*)$ *defined on the character group* G^* *of* G.

§ 3. As is well known[1], for any locally compact abelian group G^*, there exists a compact abelian group $\bar{G}^*$ and a continuous isomorphism $g^{*\prime}=\varphi^*(g^*)$ of G^* onto a subgroup $G^{*\prime}$ of $\bar{G}^*$ which is dense in $\bar{G}^*$ with the following property: for any complex-valued Bohr almost periodic function $f^*(g^*)$ defined on G^*, there exists a complex-valued continuous function $\bar{f}^*(\bar{g}^*)$ defined on $\bar{G}^*$ such that $\bar{f}^*(\varphi^*(g^*))=f^*(g^*)$ for all $g^*\in G^*$. This group $\bar{G}^*$ is called the *universal Bohr compactification* of G^*. Since conversely every complex-valued continuous (and hence Bohr almost periodic) function $\bar{f}^*(\bar{g}^*)$ defined on $\bar{G}^*$ determines a complex-valued Bohr almost periodic function $f^*(g^*)=\bar{f}^*(\varphi^*(g^*))$ on G^*, so we see that the normed ring $BAP(G^*)$ is isometrically isomorphic with the normed ring $C(\bar{G}^*)$ $=BAP(\bar{G}^*)$ of all complex-valued continuous functions $\bar{f}^*(\bar{g}^*)$ defined on $\bar{G}^*$.

On the other hand, it is also known[2] that if G^* is the character group of a locally compact abelian group G, then the universal Bohr compactification $\bar{G}^*$ of G^* is topologically isomorphic with the compact character group $G^{(d)*}$ of a discrete abelian group $G^{(d)}$ which is algebraically isomorphic with G. In fact, $G^{*\prime}$ is first obtained by introducing on G^* a weaker uniform structure $(G^*, V_\gamma^*, \Gamma)$, where

$$(10)\qquad V_\gamma^*=\{(g^*,h^*)\,|\,|(a_p,g^*)-(a_p,h^*)|<\varepsilon,\ p=1,\dots,k\}$$

$$(11)\quad \Gamma=\Big\{\gamma=\{a_1,\dots,a_k;\varepsilon\}\,|\,\{a_1,\dots,a_k\}\subseteqq G,\ k=1,2,\dots;\ \varepsilon>0\Big\},$$

with respect to which $G^{*\prime}$ is totally bounded, and then $\bar{G}^*$ is obtained by taking the completion of $G^{*\prime}$. Since $G^{(d)*}$ is topologically isomorphic with the group $\mathfrak{X}(G)$ of all algebraic (=not necessarily continuous) characters $\chi(a)$ defined on G with the usual weak topology, so we see

Theorem 2. The normed ring $\mathfrak{R}(G)$ *of a locally compact abelian group* G *is isometrically isomorphic with the normed ring* $C(\mathfrak{X}(G))=C(G^{(d)*})$ *of all complex-valued continuous functions defined on a compact abelian group* $\mathfrak{X}(G)=G^{(d)*}$, *where we mean by* $\mathfrak{X}(G)$ *the group of all algebraic (=not necessarily continuous) characters* $\chi(a)$ *defined on* G *with the usual weak topology, i.e. a compact abelian group topologically isomorphic with the character group* $G^{(d)*}$ *of a discrete abelian group* $G^{(d)}$ *which is algebraically isomorphic with* G.

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1) T. Tannaka, Über den Dualitätssatz der nichtkommutativen topologischen Gruppen, Tôhoku Math. Journ. **45** (1938), 1-12.

2) H. Anzai and S. Kakutani, Bohr compactifications of a locally compact abelian group, to appear in Proc. **19** (1943).

§ 4. From Theorem 2 follows that there exists a one-to-one correspondence between the family $\mathfrak{M}(G)$ of all maximal ideals M of $\mathfrak{R}(G)$ and the family $\mathfrak{X}(G)$ of all algebraic characters $\chi(a)$ defined on G. We shall now determine the precise way in which the correspondence is given. As we have seen in § 2, every maximal ideal M of $\mathfrak{R}(G)$ determines an algebraic character $\chi_M(a)$ on G by means of the relation: $\varphi_M(U_a)=\exp\left(2\pi i\chi_M(a)\right)$, where $\varphi_M(B)$ is a continuous natural homomorphism of the ring $\mathfrak{R}(G)$ onto the ring of complex numbers which is determined by M. Conversely, let $\chi_0(a)$ be an arbitrary algebraic character on G. Then

$$(12) \qquad A=\sum_{p=1}^{k}\alpha_p U_{a_p} \rightarrow \varphi_0(A)=\sum_{p=1}^{k}\alpha_p \exp\left(2\pi i\chi_0(a_p)\right)$$

determines an algebraic representation of $\mathfrak{A}(G)$ by complex numbers. From the fact we have observed in the proof of Theorem 1, we see that this representation may be considered as an algebraic representation $f_A^* \rightarrow \varphi_0(f_A^*)$ of $FLC(G^*)$ by complex numbers given by

$$(13) \quad f_A^*(g^*)=\sum_{p=1}^{k}\alpha_p \exp\left(2\pi i(a_p, g^*)\right) \rightarrow \varphi_0(f_A^*)=f_A^*(\chi_0)$$
$$=\sum_{p=1}^{k}\alpha_p \exp\left(2\pi i\chi_0(a_p)\right)=\sum_{p=1}^{k}\alpha_p \exp\left(2\pi i(a_p, \chi_0)\right).$$

This shows that $\varphi_0(f_A^*)$ is obtained first by extending each function $f_A^*(g^*)$ on G^* to a continuous function $f_A^*(\chi)=\sum_{p=1}^{k}\alpha_p \exp\left(2\pi i\chi(a_p)\right)$ $=\sum_{p=1}^{k}\alpha_p \exp\left(2\pi i(a_p, \chi)\right)$ on $\bar{G}^*=\mathfrak{X}(G)$, and then by taking the value of $f_A^*(\chi)$ at a particular point $\chi_0 \in \bar{G}^*=\mathfrak{X}(G)$. Since G^* is dense in $\bar{G}^*$ $=\mathfrak{X}(G)$, so we see

$$(14) \qquad |\varphi_0(f_A^*)| \leqq \sup_{\chi\in\mathfrak{X}(G)}|f_A^*(\chi)|=\sup_{g^*\in G^*}|f^*(g^*)|,$$

or equivalently

$$(15) \qquad |\varphi_0(A)| \leqq |||A||| =\|f_A^*\|.$$

Thus it is possible to extend $\varphi_0(A)$ from $\mathfrak{A}(G)$ to $\mathfrak{R}(G)$ (i. e. to extend $\varphi_0(f_A^*)$ from $FLC(G^*)$ to $BAP(G^*)$), and thus we obtain a continuous representation $B \rightarrow \bar{\varphi}_0(B)$ of $\mathfrak{R}(G)$ (i. e. a continuous representation $f^* \rightarrow \bar{\varphi}_0(f^*)=f^*(\chi_0)$ of $BAP(G^*)=C(G^*)=C\left(\mathfrak{X}(G)\right)=C(G^{(d)*})$ which is obtained by taking the value of $f^*(\chi)$ at $\chi=\chi_0$). If we now put $M=$ $\{B\,|\,\bar{\varphi}_0(B)=0\}$, then it is clear that $\chi_M(a)=\chi_0(a)$ for all $a\in G$. Thus

Theorem 3. There exists a one-to-one correspondence between the family $\mathfrak{M}(G)$ of all maximal ideals M of $\mathfrak{R}(G)$ and the family $\mathfrak{X}(G)$ of all algebraic (=not necessarily continuous) characters $\chi_M(a)$ on G given by the following relation:

$$(16) \qquad \varphi_M(U_a)=\exp\left(2\pi i\chi_M(a)\right),$$

where $B \rightarrow \varphi_M(B)$ is a continuous natural homomorphism of $\mathfrak{R}(G)$ onto *the ring of complex numbers determined by M. More precisely, for any maximal ideal M of* $\mathfrak{R}(G)$, *the relation* (16) *determines an algebraic character* $\chi_M(a)$ *on G, and conversely, for any algebraic character* $\chi_0(a)$ *on G, the algebraic representation* $A \rightarrow \varphi_0(A)$ of $\mathfrak{A}(G)$ *by complex numbers which is given by* (12) *can be uniquely extended to a continuous representation* $B \rightarrow \bar{\varphi}_0(B)$ *of* $\mathfrak{R}(G)$ *by complex numbers such that the maximal ideal* $M = \{B \mid \bar{\varphi}_0(B) = 0\}$ *determined by* $\bar{\varphi}_0(B)$ *gives an algebraic character* $\chi_M(a)$ *which satisfies* $\chi_M(a) = \chi_0(a)$ *for all* $a \in G$.