95. *Note on Free Topological Groups.*

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The notion of free topological groups has been introduced by A. Markoff[1]. The present note is to give two remarks concerning it. The first is to show that free topological groups are always maximally almost periodic; the case of discrete free groups being well-known[2] And this fact, combined with an additional observation, shows us the embeddability of any completely regular space into a totally bounded topological group as a closed subspace.

Our second remark is concerned with a refinement of the notion of free topological groups, namely, that of uniform free topological groups generated by a uniform space. It contains Markoff's free group as its special case where the completely regular space is considered under its finest uniformity.

1. *Maximally almost periodicity of free topological groups.* Let R be a completely regular space. The free topological group F generated by R is characterized by the properties:

i) R is a subspace of F,

ii) R generates F algebraically,

iii) Given a continuous mapping φ of R into any topological group, there exists a continuous homorphism Φ of F into G which is an extension of the mapping φ.

Theorem 1. The free topological group F is always maximally almost periodic.

Proof. Let g be an element of F different from the unit 1. With a certain number, say n, of elements $u_1, u_2, \dots, u_n$ from R, g is expressed in a from

$$g=u_{i_1}^{\varepsilon_1}u_{i_2}^{\varepsilon_2}\cdots u_{i_m}^{\varepsilon_m} \qquad (\varepsilon_k=\pm1).$$

Consider then the (algebraic, discrete) free group F_0 generated by the n elements $u_1, u_2, \dots, u_n$. There exists[3] in F_0 an invariant subgroup N_0 of a finite index and not containing g. Let $A(h)$ $(h\in F_0)$ be a faithful unitary representation of the finite factor group F_0/N_0, and put for the sake of simplicity

$$A_1=A(u_1),\ A_2=A(u_2),\ \dots,\ A_n=A(u_n).$$

Since the group[4] $\mathfrak{U}$ of unitary matrices, of the same degree as the representation $A(h)$, is connected, there exist in $\mathfrak{U}$ n continuous paths π_i

1) A. Markoff, On free topological groups, C. R. URSS. **31** (1941).

2) J. v. Neumann-E. P. Wigner, Minimally almost periodic groups, Ann. Math. **41** (1940); V. L. Nisnewitsch, Über Gruppen, die durch Matrizen über einem kommutativen Feld isomorph darstellbar sind, Rec. Math. **51** (1940); K. Iwasawa, Iso-Sugaku **4** (1942).

3) See 2).

4) Topologized as usual.

connecting a certain point (that is a matrix) A_0 in $\mathfrak{U}$ with the n points A_i, respectively. Let π_i be

$$\pi_i : A_i(\tau) \quad (0 \leqq \tau \leqq 1) \quad \big(A_i(0)=A_0,\ A_i(1)=A_i\big).$$

Choose in R, on the other hand, n neighbourhoods V_i of the n points $u_1, u_2, \dots, u_n$, respectively, which do not intersect each other, and let $f_i(x)$ be, correspondingly, n continuous functions on R such that $0 \leqq f_i(x) \leqq 1$ and

$$f_i(u_i)=1, \qquad f_i(x)=0 \text{ outside of } V_i.$$

Put then

$$A(x)=\begin{cases} A_0 \text{ when } x \text{ belongs to none of } V_i \\ A_i\big(f_i(x)\big) \text{ when } x \in V_i. \end{cases}$$

(Since this gives $A(u_i)=A_i$, the definition does not contradict our original significance of $A(u_i)$). It is then obvious that

$$\varphi : \qquad x \to A(x)$$

is a continuous mapping of R into $\mathfrak{U}$. There is, therefore, a continuous homomorphism Φ of F into $\mathfrak{U}$, which is an extension of φ. The mapping Φ is nothing but a continuous unitary representation of F, and we have $\Phi(u_i)=A_i=A(u_i)$, whence

$$\Phi(g)=A_{i_1}^{\epsilon_1}A_{i_2}^{\epsilon_2}\dots A_{i_m}^{\epsilon_m}=A(u_{i_1})^{\epsilon_1}A(u_{i_2})^{\epsilon_2}\dots A(u_{i_m})^{\epsilon_m}=A(g)\neq E.$$

Thus there exists for each $g \neq 1$ in F a continuous unitary representation of F "telling g apart from 1", and this proves our theorem.

Now, the totality of (continuous) almost periodic functions on F induces as usual a new (uniform) topology in F, under which F is totally bounded and is again a topological group satisfying Hausdorff's axiom of separation, *in virtue of Theorem 1.* We shall denote the group F by F^* when considered under this new topology. The topology is in general weaker than the original one of F, yet induces in the subset R the same topology. Indeed we have

Theorem 2. The totally bounded group F^* has the properties:

i) *F^* contains R as a subspace,*

ii) *R generates F^* algebraically,*

iii) *Given a continuous mapping φ^* of R into any totally bounded topological group G^*, there exists a continuous homomorphism Φ^* of F^* into G^* which is an extension of φ^*.* (Thus F^* may be called by right a *free totally bounded group*). *F^* is characterized uniquely up to topological isomorphism by these properties.*

Proof. Let u be a point in R, and V its neighbourhood. Let $f(x)$ be a continuous function on R equal to 1 at u and O outside of V, and put $\alpha(x)=\exp.\big(i\pi f(x)\big)$. $\alpha(x)$ may be extended to a continuous unitary representation (of degree 1) of F, and the set $\epsilon(h; |\alpha(h)-1| = |\alpha(h)-\alpha(u)| < 2)$ defines a neighbourhood of u in F^*. Its intersection with R is contained in V. This shows that the topology of F^*

restricted in R is stronger than, whence coincides with, the original topology of R, and i*) is proved. ii*) is evident, and iii*) is also easy to see. Further we have

Theorem 3. R is closed (not only in F but also) in F^*.

Proof. Let g be an element of F not belonging to R, and $u_1, u_2, \dots, u_n, F_0$ have the same significances as before.[1] Let N_0 be an invariant subgroup in F_0 of finite index such that g is congruent to none of u_i mod. N_0, and $A(h)$ ($h \in F_0$) be a faithful unitary representation of F_0/N_0. We extend it to a continuous unitary representation of F in the same manner as before, but that we take care so as the common starting point (matrix) A_0 be different from $A(g)$ and the paths π_i connecting it with $A(u_i)$ do not go through a certain neighbourhood of $A(g)$; this last is possible if only $A(h)$ is of higher degree than 1 (whence the dimension of $\mathfrak{U}$ is greater than 1). Then g has a positive distance from R with respect to the representation $A(h)$ of F. This proves our assertion.

Corollary. A completely regular (resp. compact) space can always be embedded in a totally bounded (resp. compact) topological group as a closed subspace. There are totally bounded groups which are not normal.

2. *Uniform free topological groups.* Consider a topological group G, and let $\{V(1)\}$ be a system of basic neighbourhoods of its unit element 1. G has a uniform topology with the sets V of (a, b) such that $ab^{-1} \in V(1)$ as a system of basic "entourages". We shall call it the right uniformity of G. Now, the topology in G may be given by a multinorm $\mathfrak{M}$, to be in accordance with Markoff's terminology; a norm N of a group Q is a non-negative function on G such that

$$N(1)=0, \qquad N(ab^{-1}) \leqq N(a)+N(b),$$

and a multinorm $\mathfrak{M}$ is a set $\mathfrak{M}=\{N\}$ of norms such that

1) $N, N_2 \in \mathfrak{M}$ implies $N_1+N_2 \in \mathfrak{M}$,

2) $N \in \mathfrak{M}$ implies $N_{(p)} \in \mathfrak{M}$ for every p in G, where $N_{(p)}(a)=N(p^{-1}ap)$,

3) if $N(a)=0$ for every $N \in \mathfrak{M}$ then $a=1$.

The sets $V_N(1)$ of a with $N(a) < 1$ form a system of basic neighbourhoods of 1.

On taking a norm $N \in \mathfrak{W}$, consider a family of functions

$$f_c(a)=f_{N,c}(a)=N(ac)$$

where c runs over G. Then this family $\{f_c(a)\}$ is uniformly equicontinuous on G, with respect to the right uniformity in G. For, given $\varepsilon > 0$, let $1/m \leqq \varepsilon$ and consider the entourage V_{mN} consisting of (a, b) with $mN(ab^{-1}) < 1$. Then for $(a, b) \in V_{mN}$

$$f_c(a)=N(ac)=N(ab^{-1}bc) \leqq N(ab^{-1})+N(bc) < \varepsilon+f_c(b).$$

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1) Only that here g may be the unit element 1.

Similarly $f_c(b) < \varepsilon + f_c(a)$. Hence $|f_c(a) - f_c(b)| < \varepsilon$, and this proves the assertion.

If R is a subset of G, the right uniformity of G induces a uniformity on R, which we shall denote by $\mathfrak{T}_0$. Let $\mathfrak{T}$ be an arbitrary second uniformity in R. From the above observations it is evident that if $\mathfrak{T}$ is finer than $\mathfrak{T}_0$ then the family of functions $\{f_c(x)\}$ is, for any $N \in \mathfrak{M}$, uniformly equi-continuous on R with respect to $\mathfrak{T}$. But the converse is the case too. Namely, on assuming the uniform equicontinuity of $\{f_c(x)\}$ with respect to $\mathfrak{T}$, we take an entourage W, depending on N, of $\mathfrak{T}$ such that $(x, y) \in W$ implies $|N(xc) - N(yc)| = |f_c(x) - f_c(y)| < 1$. When we put $c = y$, this gives $N(xy^{-1}) < 1$, or, $(x, y) \in V_N$. Thus $\mathfrak{T}$ is finer than $\mathfrak{T}_0$.

After these preliminaries we prove

Theorem 4. *Let R be a uniform space (satisfying Hausdorff's separation axiom). There exists a topological group F such that*

I) *R is a subspace of F and the right uniformity of F induces on R the original uniformity of R,*

II) *R generates F algebraically,*

III) *Given a uniformly continuous mapping φ of R into any topological group G with respect to its right uniformity, there exists a (necessarily uniformly) continuous homomorphism Φ of F into G which is an extension of φ.*

F is determined uniquely up to topological isomorphism leaving each point of R fixed. Furthermore, R is closed in F.

Proof. Let R be a uniform space, and denote its uniformity by $\mathfrak{T}$. Let F be the (algebraic) free group generated by the set R, and consider the totality $\mathfrak{M}$ of norms in F such that

$$(*) \begin{cases} \text{for any } p \text{ in } F \text{ the family of functions } N(p^{-1}xpq),\ q \text{ running} \\ \quad \text{over } F, \text{ is uniformly equi-continuous on } R \text{ in the sense of } \mathfrak{T}. \end{cases}$$

We show that $\mathfrak{M}$ forms a multinorm in F. The properties 1), 2) are evident. To verify 3), let[1] $g = u_{i_1}^{\varepsilon_1} u_{i_2}^{\varepsilon_2} \dots u_{i_m}^{\varepsilon_m}$ $(u_i \in R)$ be an element in F different from 1, and let the groups F_0, N_0, the representation $A(h)$ $(h \in F_0)$ have the same significances as in the proof of Theorem 1. Then, taking a point A_0 and n (necessarily uniformly) continuous paths π_i in the unitary group $\mathfrak{U}$, we can as before extend $A(h)$ to a unitary representation $\Phi(h)$ of the whole F such that it is uniformly continuous on R. Let $N_{\mathfrak{U}}$ be the usual invariant norm (=the distance from the unit matrix in the sense of invariant metric) of the unitary group $\mathfrak{U}$, and put $N(h) = N_{\mathfrak{U}}(\Phi(h))$ for $h \in F$. Then this norm N satisfies (*), whence belongs to $\mathfrak{M}$, and it separates g from 1. This proves 3) for $\mathfrak{M}$, and $\mathfrak{M}$ forms a multinorm. We topologize F by means of this multinorm $\mathfrak{M}$, and want to show that it possesses the properties of our theorem. That the original uniformity $\mathfrak{T}$ in R is finer than the one induced by the right uniformity of F, which we shall denote by $\mathfrak{T}_0$, follows from our preliminary observations and the property (*) of $\mathfrak{M}$. To see the converse, let W be any entourage of R in the sense

1) For the purpose of establishing Theorem 4 only, there is a shorter construction than ours.

of $\mathfrak{T}$, and $\rho(x, y)$ be a quasi-metric in R such that $\rho(x, y) \leq 1$ for every pair x, y and that $\rho(x, y) < 1/2$, for instance, implies $(x, y) \in W$. Let x_0 be an arbitrary fixed point of R. We put $\psi(x) = \exp.\big(i\pi\rho(x, x_0)\big)$, and extend the mapping $x \to \psi(x)$ to a homomorphism $h \to \Psi(h)$ of F onto the group of complex numbers of absolute value 1. Then the function $N(h) = |\Psi(h) - 1|$ forms a norm in F, and in fact belongs to $\mathfrak{M}$ since it satisfies (*). And here $N(xy^{-1}) < 1 \ (x, y \in R)$ implies $\rho(x, y) < 1/2$ whence $(x, y) \in W$. This shows that $\mathfrak{T}_0$ is finer than $\mathfrak{T}$. Thus $\mathfrak{T}_0$ and $\mathfrak{T}$ are equivalent to each other, which proves I). The property II) of F is evident. Also III) follows readily from our construction of the topology in F. The existence of a *uniform free topological group*, as we shall call it, is thus established, and its uniqueness up to topological isomorphism is obvious. The closedness of R in F, further, may be seen similarly as in Theorem 3.

Now, let us observe that when a completely regular space R is considered as a uniform space under the finest uniformity belonging to its topology, the above difined uniform free topological group becomes nothing but Markoff's usual free topological group. Indeed, the finest uniformity $\mathfrak{T}$ is characterized by that any continuous mapping of R into a second uniform space is uniformly continuous. Hence, if $\mathfrak{T}'$ denotes the uniformity induced in R by the right uniformity of the usual free topological group generated by R, the identity transformation in R is a uniformly continuous 1–1 mapping of the uniform space R in the sense of $\mathfrak{T}$ onto the uniform space R in the sense of $\mathfrak{T}'$. Because of the characteristic property of the uniform free topological group, the identity transformation of F is, therefore, a continuous isomorphism from the uniform free topological group F onto the usual one F. But the transformation is continuous also in the inverse direction in virtue of the definition of the usual free topological group. This proves our assertion.

Further, one verifies as before that also uniform free topological groups are maximally almost periodic.

3. In connection with uniform topology and uniform continuity, we note finally that the notion of (*uniform or mere*) *free uniform-topological groups* may be introduced in like manner. Here we mean by a uniform-topological group a topological group in which group operations are uniformly continuous with respect to its right uniformity; the same is the case then with respect to the left uniformity. And, (uniform or mere) free uniform-topological groups are defined by replacing " topological group " by " uniform-topological group " in the definition of (uniform or mere) free topological groups. Uniform-topological groups are characterized by that they possess multinorms consisting of invariant norms. Thus the existence of free ones can be established similarly as above by taking into consideration invariant norms only; observe that the norms $N_{\mathfrak{U}}\big(\Phi(h)\big)$ and $|\Psi(h) - 1|$ used above are all invariant ones.