38. *On Certain Spaces Admitting Concircular Transformations.*

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§0. *Introduction.* One of the present authors had studied the conformal transformations

(0.1) $$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$$

of Riemannian metrics which change any Riemannian geodesic circle

(0.2) $$\frac{\delta^3 x^\lambda}{ds^3}+\frac{dx^\lambda}{ds}g_{\mu\nu}\frac{\delta^2 x^\mu}{ds^2}\frac{\delta^2 x^\nu}{ds^2}=0$$

into a Riemannian geodesic circle, and called such transformations concircular transformations.[1]

In order that the conformal transformation (0.1) be a concircular one, it is necessary and sufficient that the function ρ satisfies the differential equations

(0.3) $$\rho_{\mu\nu}\equiv\rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\beta\gamma}\rho_\beta\rho_\gamma g_{\mu\nu}=\phi g_{\mu\nu},$$

where $\rho_\mu=(\log\rho)_{;\mu}$ and the semi-colon denotes the covariant differentiation with respect to the Christoffel symbols $\{^{\lambda}_{\mu\nu}\}$, ϕ being a certain scalar.

If we put

(0.4) $$\sigma=\frac{1}{\rho},$$

the condition (0.3) may also be written as

(0.5) $$\sigma_{\mu;\nu}=\alpha g_{\mu\nu},$$

where $\sigma_\mu=\sigma_{;\mu}$ and α is a certain scalar.

If the partial differential equations (0.3) or (0.5) admit a solution, the family of hypersurfaces defined by ρ=constant or σ=constant are totally umbilical and their orthogonal trajectories are geodesic Ricci curves.

Conversely, if a Riemannian space contains a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesic Ricci curves, the space admits a concircular transformation.

In the present note, we shall study the spaces which admit the concircular transformation and satisfy some additional conditions.

§1. We shall first consider a Riemannian space which admits a

(1) K. Yano : Concircular Geometry, I, II, III, IV, V. Proc. 16(1940), 195-200 ; 354-360 ; 442-445 ; 505-511 ; 18 (1942), 446-451.

concircular transformation and which is conformally flat.

As our space admits a concircular transformation, there must exist a function σ which satisfies

$$\sigma_{\mu;\nu}=\alpha g_{\mu\nu}, \tag{1.1}$$

where $\sigma_\mu=\sigma_{;\mu}$ and α is a certain function of coordinates. Thus, differentiating the quantity $g^{\beta\gamma}\sigma_\beta\sigma_\gamma$ covariantly and taking account of (1.1), we find

$$(g^{\beta\gamma}\sigma_\beta\sigma_\gamma)_{;\nu}=2\alpha\sigma_\nu,$$

from which we can conclude that the quantities $g^{\beta\gamma}\sigma_\beta\sigma_\gamma$ and α are both functions of σ alone. Thus we can put

$$g^{\beta\gamma}\sigma_\beta\sigma_\gamma=\varphi(\sigma), \qquad \alpha=\alpha(\sigma). \tag{1.2}$$

Next, substituting (1.1) into the Ricci identities

$$-\sigma_\lambda R^\lambda{}_{\cdot\mu\nu\omega}=\sigma_{\mu;\nu;\omega}-\sigma_{\mu;\omega;\nu},$$

we find

$$-\sigma_\lambda R^{\mu\nu\omega}_{\cdot}=\alpha'(g_{\mu\nu}\sigma_\omega-g_{\mu\omega}\sigma_\nu). \tag{1.3}$$

Multiplying this equation by $g^{\mu\nu}$ and summing up with respect to the indices μ and ν, we find

$$-\sigma_\lambda R^\lambda_{\cdot\,\omega}=(n-1)\alpha'\sigma_\omega, \tag{1.4}$$

which shows that the direction $\sigma^\lambda(=g^{\lambda\alpha}\sigma_\alpha)$ is a Ricci direction, $R^\lambda_{\cdot\,\omega}$ being the mixed components of the Ricci tensor $R_{\mu\nu}(=R^\alpha_{\cdot\,\mu\nu\alpha})$.

Now, the space being supposed to be conformally flat, we have

$$R^\lambda_{\cdot\,\mu\nu\omega}=\frac{1}{n-2}\left(R_{\mu\nu}\delta^\lambda_\omega-R_{\mu\omega}\delta^\lambda_\nu+g_{\mu\nu}R_{\cdot\omega}^{\lambda}-g_{\mu\omega}R_{\cdot\nu}^{\lambda}\right)-\frac{R}{(n-1)(n-2)}\left(g_{\mu\nu}\delta^\lambda_\omega-g_{\mu\omega}\delta^\lambda_\nu\right).$$

Substituting this into (1.3) and taking account of (1.4), we find

$$-\frac{1}{n-2}\left(R_{\mu\nu}\sigma_\omega-R_{\mu\omega}\sigma_\nu-(n-1)\alpha' g_{\mu\nu}\sigma_\omega+(n-1)\alpha' g_{\mu\omega}\sigma_\nu\right)$$
$$+\frac{R}{(n-1)(n-2)}(g_{\mu\nu}\sigma_\omega-g_{\mu\omega}\sigma_\nu)=\alpha'(g_{\mu\nu}\sigma_\omega-g_{\mu\omega}\sigma_\nu),$$

or

$$\left[-\frac{R_{\mu\nu}}{n-2}+\frac{Rg_{\mu\nu}}{(n-1)(n-2)}+\frac{\alpha' g_{\mu\nu}}{n-2}\right]\sigma_\omega=\left[-\frac{R_{\mu\omega}}{n-2}+\frac{Rg_{\mu\omega}}{(n-1)(n-2)}+\frac{\alpha' g_{\mu\omega}}{n-2}\right]\sigma_\lambda.$$

From this equation, we can conclude that

$$-\frac{R_{\mu\nu}}{n-2}+\left[\frac{R}{(n-1)(n-2)}+\frac{\alpha'}{n-2}\right]g_{\mu\nu}=\psi\sigma_\mu\sigma_\nu, \tag{1.5}$$

where ψ is a certain scalar.

Multiplying this equation by $g_{\mu\nu}$ and summing up for the indices μ and ν, we obtain

$$\frac{R}{(n-1)(n-2)}+\frac{n\alpha'}{n-2}=\psi g^{\mu\nu}\sigma_\mu\sigma_\nu. \tag{1.6}$$

On the other hand, we have, from (1.5),

$$-\frac{R^\lambda{}_\nu}{n-2}+\left[\frac{R}{(n-1)(n-2)}+\frac{\alpha'}{n-2}\right]\delta^\lambda_\nu=\psi\sigma^\lambda\sigma_\nu.$$

Differentiating this covariantly with respect to x^λ, contracting with respect to λ, and taking account of (1.1) and of the identities

$$R^{\lambda}_{.\ \nu;\lambda}=\frac{1}{2}R_{;\nu}\ ,$$

we find

$$-\frac{R_{;\nu}}{2(n-2)}+\frac{R_{;\nu}}{(n-1)(n-2)}+\frac{a''\sigma_\nu}{n-2}=\psi_{;\lambda}\sigma^\lambda\sigma_\nu+(n+1)a\psi\sigma_\nu,$$

or

$$-\frac{(n-3)R_{;\nu}}{2(n-1)(n-2)}=\left(\psi_{;\lambda}\sigma^\lambda+(n+1)a\psi-\frac{a''}{n-2}\right)\sigma_\nu.$$

Thus, we can see that, when $n>3$, R is also a function of σ alone. Thus the equation (1.2) and (1.6) show that ψ is also a function of σ alone. Thus we can put

$$R=R(\sigma)\,,\quad \psi=\psi(\sigma)\,.$$

Consquently, the equation (1.5) gives

$$\Pi_{\mu\nu}\equiv-\frac{R_{\mu\nu}}{n-2}+\frac{Rg_{\mu\nu}}{2(n-1)(n-2)}=f(\sigma)g_{\mu\nu}+\psi(\sigma)\sigma_\mu\sigma_\nu. \tag{1.8}$$

The space which is conformally flat and whose tensor $\Pi_{\mu\nu}$ has the form (1.8) being a subprojective one, we have the

Theorem 1.1. *The $n(>3)$-dimensional Riemannian space which admits a concircular transformation and is conformally flat is a subprojective space of Kagan.*

Conversely, for a subprojective space, we have

$$C^{\lambda}_{.\ \mu\nu\omega}\equiv R^{\lambda}_{.\ \mu\nu\omega}+\Pi_{\mu\nu}\delta^\lambda_\omega-\Pi_{\mu\omega}\delta^\lambda_\nu+g_{\mu\nu}\Pi^{\lambda}_{.\ \omega}-g_{\mu\omega}\Pi^{\lambda}_{.\ \nu}=0, \tag{1.9}$$

$$C_{\mu\nu\omega}\equiv\Pi_{\mu\nu;\omega}-\Pi_{\mu\omega;\nu}=0, \tag{1.10}$$

and (1.8). Substituting (1.8) into (1.10), we find

$$f'(g_{\mu\nu}\sigma_\omega-g_{\mu\omega}\sigma_\nu)+\psi(\sigma_{\mu;\omega}\sigma_\nu-\sigma_{\mu;\nu}\sigma_\omega)=0.$$

Multiplying this equation by $g_{\mu\nu}$ and contracting with respect to μ and ν, we have

$$(n-1)f'\sigma_\omega+\frac{1}{2}\psi(g^{\beta\gamma}\sigma_\beta\sigma_\gamma)_{;\omega}-\psi g^{\mu\nu}\sigma_{\mu;\nu}\sigma_\omega=0,$$

which shows that $g^{\beta\gamma}\sigma_\beta\sigma_\gamma$ is a function of σ alone, that is to say,

$$g^{\mu\nu}\sigma_\mu\sigma_\nu=\psi(\sigma)\,. \tag{1.11}$$

Next, multiplying (1.8) by $g^{\mu\nu}$ and contracting, we find

$$-\frac{R}{2(n-1)}=nf(\sigma)+\psi(\sigma)g^{\mu\nu}\sigma_\mu\sigma_\nu,$$

from which we can see that R is a function of σ alone, that is,

$$R=R(\sigma)\,. \tag{1.12}$$

Now, from (1.9), we have

$$-\sigma_\lambda R^{\lambda}_{.\ \mu\nu\omega}=\Pi_{\mu\nu}\sigma_\omega-\Pi_{\mu\omega}\sigma_\nu+g_{\mu\nu}\sigma^\lambda\Pi_{\lambda\omega}-g_{\mu\omega}\sigma^\lambda\Pi_{\lambda\nu},$$

from which, we have, taking account of (1.8),

$$-\sigma_\lambda R^{\lambda}_{.\ \mu\nu\omega}=\left(2f(\sigma)+\psi(\sigma)g^{\beta\gamma}\sigma_\beta\sigma_\gamma\right)(g_{\mu\nu}\sigma_\omega-g_{\mu\omega}\sigma_\nu)\,,$$

or

$$-\sigma_\lambda R^{\lambda}_{\cdot\,\mu\nu\omega}=\Big(2f(\sigma)+\psi(\sigma)\varphi(\sigma)\Big)\;(g_{\mu\nu}\sigma_\omega-g_{\mu\omega}\sigma_\nu). \tag{1.13}$$

The equation (1.13) shows that the differential equations

$$\sigma_{\mu;\nu}=ag_{\mu\nu}$$

are integrable. Thus the space admits a concircular transformation and consequently we have the

Theorem 1.2. *Subprojective space of Kagan admits a concircular transformation.*

§2. If a Riemannian space admits a concircular transformation there exists a function σ such that

$$\sigma_{\mu;\nu}=ag_{\mu\nu}, \tag{2.1}$$

and we know that the hypersurfaces defined by $\sigma=$const. are all totally umbilical and their orthogonal trajectories are geodesic Ricci curves.

We shall represent one of these hypersurfaces by parametric equations

$$x^\lambda=x^\lambda(u^i) \qquad (i,j,k,\cdots=1,2,\cdots n-1), \tag{2.2}$$

then we have

$$\sigma_\lambda B_i^{\cdot\,\lambda}=0, \tag{2.3}$$

where

$$B_i^{\cdot\,\lambda}=\frac{\partial x^\lambda}{\partial u^i}, \tag{2.4}$$

that is, the vector σ_λ is normal to the hypersurface. Thus denoting by B^λ the unit vector normal to the hypersurface, we have

$$\sigma_\lambda=\sqrt{g^{\beta\gamma}\sigma_\beta\sigma_\gamma}\,B_\lambda.$$

Denoting the first and the second fundamental tensors of the hypersurface by g_{ij} and H_{ij} respectively and the curvature tensor by $R^{i}_{\cdot\,jkl}$, the equations of Gauss for the hypersurface may be written as

$$R^{i}_{\cdot\,jkl}=B^{i\mu\nu\omega}_{\lambda jkl}R^{\lambda}_{\cdot\,\mu\nu\omega}+H_{jk}H^{i}_{\cdot\,l}-H_{jl}H^{i}_{\cdot\,k}, \tag{2.5}$$

where

$$B^{i\mu\nu\omega}_{\lambda jkl}=B^{i}_{\cdot\,\lambda}B_j^{\cdot\,\mu}B_k^{\cdot\,\nu}B_l^{\cdot\,\omega} \text{ and } B^{i}_{\cdot\,\lambda}=g^{ij}g_{\lambda\mu}B_j^{\cdot\,\mu}.$$

But, the hypersurface being totally umbilical, we have

$$H_{jk}=Hg_{jk}.$$

Thus the equations (2.5) become

$$R^{i}_{\cdot\,jkl}=B^{i\mu\nu\omega}_{\lambda jkl}R^{\lambda}_{\cdot\,\mu\nu\omega}+H^2(g_{jk}\delta^i_l-g_{jl}\delta^i_k). \tag{2.6}$$

In these equations, contracting with respect to i and l and taking account of the relation

$$B^{i}_{\cdot\,\lambda}B_i^{\cdot\,\omega}=\delta^\omega_\lambda-B_\lambda B^\omega,$$

we find

$$R_{jk} \equiv R^{i}_{\cdot\, jki} = B^{\mu\nu}_{jk}(\delta^{\omega}_{\lambda} - B_{\lambda}B^{\omega}) R^{\lambda}_{\cdot\,\mu\nu\omega} + (n-2) H^2 g_{jk}.$$

Substituting (1.3) in these equations, we find

$$R_{jk} = B^{\mu\nu}_{jk} R_{\mu\nu} + \Big[(n-2) H^2 + \alpha'\Big] g_{jk}. \tag{2.7}$$

Now, we assume that, the space admitting a concircular transformation, the tensor $\Pi_{\mu\nu}$ of the space has the form

$$\Pi_{\mu\nu} = f(\sigma) g_{\mu\nu} + \psi(\sigma) \sigma_{\mu}\sigma_{\nu}, \tag{2.8}$$

Then we have

$$R_{\mu\nu} = \Big[\frac{R}{2(n-1)} - (n-2) f(\sigma)\Big] g_{\mu\nu} - (n-2)\psi(\sigma)\sigma_{\mu}\sigma_{\nu}.$$

Substituting this equation into (2.7), we find

$$R_{jk} = \Big[\frac{R}{2(n-1)} - (n-2) f(\sigma) + (n-2) H^2 + \alpha'\Big] g_{jk}.$$

Thus we have the

Theorem 2.1. *If a Riemannian space admits a concircular transformation, that is, there exists a function σ such that $\sigma_{\mu;\nu} = \alpha g_{\mu\nu}$ and the tensor $\Pi_{\mu\nu}$ of the space has the form* (2.8), *the totally umbilical hypersurfaces defined by σ=const. are Einstein spaces.*

Conversely, the space admitting a concircular transformation, that is, there existing a function σ such that $\sigma_{\mu;\nu} = \alpha g_{\mu\nu}$, if we assume that the totally umbilical hypersurfaces σ=const. are all Einstein spaces, that is,

$$R_{jk} = \gamma g_{jk},$$

we have, from (2.7),

$$B^{\mu\nu}_{jk} R_{\mu\nu} = \Big[\gamma - (n-2) H^2 - \alpha'\Big] g_{jk}.$$

Multiplying this equation by $B^{j}_{\cdot\,\beta} B^{k}_{\cdot\,\gamma}$ and taking account of

$$B^{j}_{\cdot\,\beta} B^{\cdot\,\mu}_{j} = \delta^{\mu}_{\beta} - B_{\beta}B^{\mu}, \quad g_{jk} B^{j}_{\cdot\,\beta} B^{k}_{\cdot\,\gamma} = g_{\beta\gamma} - B_{\beta}B_{\gamma},$$

and of

$$B^{\mu} R_{\mu\nu} = p B_{\nu},$$

we find

$$(\delta^{\mu}_{\beta} - B_{\beta}B^{\mu})(\delta^{\nu}_{\gamma} - B_{\gamma}B^{\nu}) R_{\mu\nu} = \Big[\gamma - (n-2) H^2 - \alpha'\Big] g_{jk} B^{j}_{\cdot\,\beta} B^{k}_{\cdot\,\gamma},$$

or

$$R_{\beta\gamma} - p B_{\beta}B_{\gamma} = \Big[\gamma - (n-2) H^2 - \alpha'\Big](g_{\beta\gamma} - B_{\beta}B_{\gamma}),$$

which shows that

$$\Pi_{\mu\nu} = f g_{\mu\nu} + \psi \sigma_{\mu}\sigma_{\nu},$$

B_{μ} and σ_{μ} being proportional. But, H^2, α', γ and p being functions of σ, we can write

$$\Pi_{\mu\nu} = f(\sigma) g_{\mu\nu} + \psi(\sigma) \sigma_{\mu}\sigma_{\nu}.$$

Thus we have the

Theorem 2.2. If a space admits a concircular transformation, that is, there exists a function σ such that $\sigma_{\mu;\nu}=ag_{\mu\nu}$ and if the totally umbilical hypersurfaces $\sigma=$const. are all Einstein spaces, then the tensor $\Pi_{\mu\nu}$ of the space has the form (2.8). ($n>3$).

If a Riemannian space which admits a concircular transformation is conformally flat, then the tensor $\Pi_{\mu\nu}$ of the space is necessarily of the form (2.8), and consequently the totally umbilical hypersurfaces $\sigma=$const. are all Einstein spaces. But totally umbilical hypersurfaces in a conformally flat space being also conformally flat,(1) these hypersurfaces are also conformally flat. These hypersurfaces being Einstein spaces and conformally flat spaces at the same time, they are spaces of constant curvature. Thus we have the

Theorem 2.3. If a Riemannian space admits a concircular transformation and is conformally flat, the hypersurfaces defined by $\sigma=$ const. are all spaces of constant curvature. ($n>3$).

§3. In this Paragraph, we shall reconsider the case in which the space admits a concircular transformation, that is, there exists a function such that

$$\sigma_{\mu;\nu}=ag_{\mu\nu} \tag{3.1}$$

and the tensor $\Pi_{\mu\nu}$ of the space has the form

$$\Pi_{\mu\nu}=f(\sigma)g_{\mu\nu}+\psi(\sigma)\sigma_\mu\sigma_\nu. \tag{3.2}$$

We know that a and $g^{\beta\gamma}\sigma_\beta\sigma_\gamma$ are both functions of σ alone. Contracting (3.2) by $g^{\mu\nu}$, we find

$$-\frac{R}{2(n-1)}=nf(\sigma)+\psi(\sigma)g^{\beta\gamma}\sigma_\beta\sigma_\gamma.$$

Differentiating this equation covariantly, and taking account of (3.1), we have

$$-\frac{R_{;\nu}}{2(n-1)}=nf'\sigma_\nu+\psi'\sigma_\nu g^{\beta\gamma}\sigma_\beta\sigma_\gamma+2\psi a\sigma_\nu. \tag{3.3}$$

On the other hand, from (3.2), we have

$$\Pi^{\lambda}_{\cdot\,\nu}=f\delta^\lambda_\nu+\psi\sigma^\lambda\sigma_\nu.$$

Differentiating this equation covariantly, and taking account of

$$\Pi^{\lambda}_{\cdot\,\nu;\lambda}=-\frac{R_{;\nu}}{2(n-1)},$$

we find

$$-\frac{R_{;\nu}}{2(n-1)}=f'\sigma_\nu+\psi'\sigma_\lambda\sigma^\lambda\sigma_\nu+(n+1)\psi a\sigma_\nu. \tag{3.4}$$

Comparing the equations (3.3) and (3.4), we find

$$f'=\psi a. \tag{3.5}$$

Thus we have

(1) K. Yano : Sur les équations de Gauss dans la géométrie conforme des espaces de Riemann, Proc. 15 (1939), 247-252.

$$\Pi_{\mu\nu;\omega}-\Pi_{\mu\omega;\nu}=f'g_{\mu\nu}\sigma_\omega+\psi'\sigma_\mu\sigma_\nu\sigma_\omega+\psi a g_{\mu\omega}\sigma_\nu+\psi a\sigma_\mu g_{\nu\omega}$$
$$-f'g_{\mu\omega}\sigma_\nu-\psi'\sigma_\mu\sigma_\omega\sigma_\nu-\psi a g_{\mu\nu}\sigma_\omega-\psi a\sigma_\mu g_{\omega\nu}$$
$$=(f'-\psi a)(g_{\mu\nu}\sigma_\omega-g_{\mu\omega}\sigma_\nu)=0,$$

and consequently we have the

Theorem 3.1. *When a space admits a concircular transformation, that is to say, there exists a function* σ *such that* $\sigma_{\mu;\nu}=ag_{\mu\nu}$ *and the space has the tensor of the form* (3.2), *then we have*

(3.6) $$\Pi_{\mu\nu;\omega}-\Pi_{\mu\omega;\nu}=0.$$

We shall now suppose that the space admits a concircular transformation and its tensor $\Pi_{\mu\nu}$ satisfies the equation (3.6).

From equation (1.4), we have

$$B_j^{\cdot\mu}B^\nu R_{\mu\nu}=0.$$

Differentiating this covariantly along the hypersurface and taking account of

$$B_j^{\cdot\mu}{}_{;k}=g_{jk}HB^\mu,\quad B^\nu{}_{;k}=-HB_k^{\cdot\nu},$$

we find

$$g_{jk}HB^\mu B^\nu R_{\mu\nu}-HB_j^{\cdot\mu}B_k^{\cdot\nu}R_{\mu\nu}+B_j^{\cdot\mu}B^\nu B_k^{\cdot\omega}R_{\mu\nu;\omega}=0.$$

Substituting (2.7) in this equation, we have

$$g_{jk}HB^\mu B^\nu R_{\mu\nu}-HR_{jk}-\Big[(n-2)H^2+\alpha'\Big]Hg_{jk}+B_j^{\cdot\mu}B^\nu B_k^{\cdot\omega}R_{\mu\nu;\omega}=0,$$

or, according to (3.6),

(3.7) $$g_{jk}HB^\mu B^\nu R_{\mu\nu}-HR_{jk}-\Big[(n-2)H^2+\alpha'\Big]Hg_{jk}$$
$$+B_j^{\cdot\mu}B^\nu B_k^{\cdot\omega}R_{\mu\omega;\nu}-\frac{R_{;\nu}B^\nu}{2(n-1)}g_{jk}=0.$$

On the other hand, considering an infinitesimal deformation

(3.8) $$\bar{x}^\lambda=x^\lambda+\sigma^\lambda\delta t,$$

we have

(3.9) $$Xg_{\mu\nu}=\sigma_{\mu;\nu}+\sigma_{\nu;\mu}=2ag_{\mu\nu},$$ (1)

and consequently the deformation defined by (3.8) is a conformal one. Substituting (3.9) in the equation

3.10) $$X\left\{{\lambda\atop \mu\nu}\right\}=\frac{1}{2}g^{\lambda\alpha}\Big[(Xg_{\alpha\mu})_{;\nu}+(Xg_{\alpha\nu})_{;\mu}-(Xg_{\mu\nu})_{;\alpha}\Big],$$

we find

(3.11) $$X\left\{{\lambda\atop \mu\nu}\right\}=\alpha'\Big[\delta^\lambda_\mu\sigma_\nu+\delta^\lambda_\nu\sigma_\mu-\sigma^\lambda g_{\mu\nu}\Big].$$

Substituting (3.11) into the equation

(3.12) $$XR^{\lambda}_{\cdot\mu\nu\omega}=\Big(X\left\{{\lambda\atop \mu\nu}\right\}\Big)_{;\omega}-\Big(X\left\{{\lambda\atop \mu\omega}\right\}\Big)_{;\nu},$$

we find

(1) See, K. Yano : Groups of transformations in generalized spaces. Akademia Press, Tokyo, 1949.

$$XR^{\lambda}_{\cdot\,\mu\nu\omega}=-\alpha''\Big[\sigma_\mu\sigma_\nu\delta^\lambda_\omega-\sigma_\mu\sigma_\omega\delta^\lambda_\nu+\sigma^\lambda g_{\mu\nu}\sigma_\omega-\sigma^\lambda g_{\mu\omega}\sigma_\nu\Big]$$
$$-\alpha'\Big[\sigma_{\mu;\nu}\delta^\lambda_\omega-\sigma_{\mu;\omega}\delta^\lambda_\nu+g_{\mu\nu}\sigma^\lambda{}_{;\omega}-g_{\mu\omega}\sigma^\lambda{}_{;\nu}\Big],$$

or

(3.13) $$XR^{\lambda}_{\cdot\,\mu\nu\omega}=-\alpha''\Big[\sigma_\mu\sigma_\nu\delta^\lambda_\omega-\sigma_\mu\sigma_\omega\delta^\lambda_\nu+\sigma^\lambda g_{\mu\nu}\sigma_\omega-\sigma^\lambda g_{\mu\omega}\sigma_\nu\Big]$$
$$-2\alpha\alpha'\Big[g_{\mu\nu}\delta^\lambda_\omega-g_{\mu\omega}\delta^\lambda_\nu\Big],$$

from which, by contraction,

(3.14) $$XR_{\mu\nu}=-\Big[2(n-1)\alpha\alpha'+\alpha''\sigma^\lambda\sigma_\lambda\Big]g_{\mu\nu}-(n-2)\alpha''\sigma_\mu\sigma_\nu.$$

The $XR_{\mu\nu}$ being given by

$$XR_{\mu\gamma}=\sigma^\alpha R_{\mu\nu;\alpha}+\sigma^\alpha{}_{;\mu}R_{\alpha\nu}+\sigma^\alpha{}_{;\nu}R_{\mu\alpha}$$
$$\sigma=^\alpha R_{\mu\nu;\alpha}+2\alpha R_{\mu\nu},$$

we have, from (3.14),

(3.15) $$\sigma^\alpha R_{\mu\nu;\alpha}=-2\alpha R_{\mu\nu}-\Big[2(n-1)\alpha\alpha'+\alpha''\sigma^\lambda\sigma_\lambda\Big]g_{\mu\nu}-(n-2)\alpha''\sigma_\mu\sigma_\nu.$$

The σ^α being proportional to B^α, we have

$$\sqrt{\sigma^\lambda\sigma_\lambda}\,B^{\mu\nu}_{jk}B^\omega R_{\mu\nu;\omega}=-2\alpha B^{\mu\nu}_{jk}R_{\mu\nu}-\Big[2(n-1)\alpha\alpha'+\alpha''\sigma^\lambda\sigma_\lambda\Big]g_{jk},$$

or

(3.16) $$\sqrt{\sigma^\lambda\sigma_\lambda}\,B^{\mu\nu}_{jk}B^\omega R_{\mu\nu;\omega}=-2\alpha R_{jk}-\Big[2n\alpha\alpha'+\alpha''\sigma^\lambda\sigma_\lambda+2\alpha(n-2)H^2\Big]g_{jk}.$$

Thus substituting (3.16) into (3.7), we find

(3.19) $$-HR_{jk}=\gamma g_{jk},\quad(\alpha\neq0),$$

because of the relation

$$\frac{\alpha}{\sqrt{\sigma^\lambda\sigma_\lambda}}=-H^{(1)}.$$

Thus we have the

Theorem 3.2. *If a space admitting a concircular transformation has the tensor $\Pi_{\mu\nu}$ such that $\Pi_{\mu\nu;\omega}-\Pi_{\mu\omega;\nu}=0$, the totally umbilical hypersurfaces which the space contains are all Einstein spaces.*

Theorem 3.3. *If a space admitting a concircular transformation has the tensor $\Pi_{\mu\nu}$ such that $\Pi_{\mu\nu;\omega}-\Pi_{\mu\omega;\nu}=0$, the tensor $\Pi_{\mu\nu}$ has the form*

$$\Pi^{\mu\nu}=f(\sigma)g_{\mu\nu}+\psi(\sigma)\sigma_\mu\sigma_\nu.$$

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(1) For the hypersurface $\sigma=$const., we have

$$\sigma_\mu B_j^{\cdot\,\mu}=0,$$
$$\sigma_{\mu;\nu}B_j^{\cdot\,\mu}B_k^{\cdot\,\nu}+\sigma_\mu H_j^{\cdot}{}_k^{\cdot\,\mu}=0,$$
$$\alpha g_{jk}+\sigma_\mu B^\mu g_{jk}=0,$$

and consequently,

$$\frac{\alpha}{\sqrt{\sigma^\lambda\sigma_\lambda}}=-H.$$