65. *Fundamental Theory of Toothed Gearing. VII.*

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We have developed in reports (V) and (VI) the general theory of spherical profile curves. In this present report (VII), we shall apply the theory to several practical spherical curves.

§ 1. Spherical profile curves of cycloidal system.

We take a small circle K_r with spherical radius λ_r as a rolling curve. In this case, however, it is not necessary to take a circle as a pitch curve K. Suppose that the curve K_r is oriented so that the radius λ_r is positive. We adopt as origin one of the two points at which the great circle passing a drawing point C inmovably connected with K_r and the center O_r intersects the perimeter of K_r, the nearer point P_0 to the point C. Take an arbitrary point P on K_r and denote by ξ the length of the arc P_0P, by φ the length of the minor arc of the great circle connecting the point P with the point C, and by θ the angle between this great circle and the tangent great circle to K_r at P. The three quantities ξ, φ and θ are all signed and moreover $\operatorname{sgn}(\varphi)=\operatorname{sgn}(\theta)$.

If we find the relation $\varphi=f(\xi)$ between φ and ξ and the relation $\varphi=g(\theta)$ between φ and θ, they are respectively the equation of the profile curve F drawn by the point C and the equation of the path of contact Γ corresponding to F. We denote by δ the length of the arc P_0C.

From the spherical triangle O_rPC we have

$$(1)\qquad \cos\varphi=\sin\lambda_r\sin(\lambda_r-\delta)\cos\left[\frac{\xi}{\sin\lambda_r}\right]+\cos\lambda_r\cos(\lambda_r-\delta),$$

that is;

when $\delta>0$

$$(1)_1\quad \varphi=f(\xi)=\cos^{-1}\left\{\sin\lambda_r\sin(\lambda_r-\delta)\cos\left[\frac{\xi}{\sin\lambda_r}\right]+\cos\lambda_r\cos(\lambda_r-\delta\right\},$$

when $\delta<0$

$$(1)_2\quad \varphi=f(\xi)=\begin{cases}-\cos^{-1}\left\{\sin\lambda_r\sin(\lambda_r-\delta)\cos\left[\frac{\xi}{\sin\lambda_r}\right]+\cos\lambda_r\cos(\lambda_r-\delta)\right\}, \\ \qquad \text{where}\quad |\xi|\leqq\sin\lambda_r\cos^{-1}\left[\frac{\tan\lambda_r}{\tan(\lambda_r-\delta)}\right], \\ \cos^{-1}\left\{\sin\lambda_r\sin(\lambda_r-\delta)\cos\left[\frac{\xi}{\sin\lambda_r}\right]+\cos\lambda_r\cos(\lambda_r-\delta)\right\}, \\ \qquad \text{where}\quad |\xi|\geqq\sin\lambda_r\cos^{-1}\left[\frac{\tan\lambda_r}{\tan(\lambda_r-\delta)}\right].\end{cases}$$

The arccosines in $(1)_1$ and $(1)_2$ express their principal values.

In particular, when $\delta=0$ —— the drawing point C exists on the perimeter of K_r ——,

$$(1)_3 \qquad \varphi=f(\xi)=\cos^{-1}\left\{\sin^2\lambda_r\cos\left[\frac{\xi}{\sin\lambda_r}\right]+\cos^2\lambda_r\right\}$$

or

$$=2\sin^{-1}\left\{\sin\lambda_r\sin\left[\frac{|\xi|}{2\sin\lambda_r}\right]\right\}.$$

Next, from the same spherical triangle O_rPC we have

$$(2) \qquad \cos(\lambda_r-\delta)=\sin\lambda_r\sin\theta\sin\varphi+\cos\lambda_r\cos\varphi,$$

namely,

$$(3) \qquad \{\cos(\lambda_r-\delta)+\cos\lambda_r\}\tan^2\frac{\varphi}{2}-2\sin\lambda_r\sin\theta\tan\frac{\varphi}{2}\{\cos(\lambda_r-\delta)-\cos\lambda_r\}=0$$

or

$$(4) \qquad \cos\left(\lambda_r-\frac{\delta}{2}\right)\cos\frac{\delta}{2}\tan^2\frac{\varphi}{2}-\sin\lambda_r\sin\theta\tan\frac{\varphi}{2}+\sin\left(\lambda_r-\frac{\delta}{2}\right)\sin\frac{\delta}{2}=0.$$

The curve represented by Equation (4), the path of contact Γ, is an arc of the small circle with the point C as center and $\lambda_r-\delta$ as spherical radius. This result may be derived directly from the characteristic property of path of contact which we discussed at the last part of the report (V) and the fact that the spherical evolute of the small circle K_r reduces to the center O_r.

Solving (4) for φ we have:

when $\delta>0$

$$(4)_1 \qquad \varphi=g(\theta)=2\tan^{-1}\left\{\frac{\sin\lambda_r\sin\theta\pm\sqrt{\sin^2\lambda_r\sin^2\theta-\sin(2\lambda_r-\delta)\sin\delta}}{\cos(\lambda_r-\delta)+\cos\lambda_r}\right\},$$

when $\delta<0$

$$(4)_2 \qquad \varphi=g(\theta)=\begin{cases} 2\tan^{-1}\left\{\dfrac{\sin\lambda_r\sin\theta+\sqrt{\sin^2\lambda_r\sin^2\theta-\sin(2\lambda_r-\delta)\sin\delta}}{\cos(\lambda_r-\delta)+\cos\lambda_r}\right\}, \\ \qquad \text{where} \quad \theta\geqq 0, \\ 2\tan^{-1}\left\{\dfrac{\sin\lambda_r\sin\theta-\sqrt{\sin^2\lambda_r\sin^2\theta-\sin(2\lambda_r-\delta)\sin\delta}}{\cos(\lambda_r-\delta)+\cos\lambda_r}\right\}, \\ \qquad \text{where} \quad \theta\leqq 0. \end{cases}$$

The arctangents in $(4)_1$ and $(4)_2$ represent their principal values.

In particular, when $\delta=0$ ——the drawing point C exists on K_r ——,

$$(4)_3 \qquad \varphi=g(\theta)=2\tan^{-1}\{\tan\lambda_r\sin\theta\}, \qquad \text{where} \quad \theta\geqq 0.$$

$(4)_3$ is, at the same time, the equation of the rolling curve K_r itself.

Next, denote the natural equations of the pitch curves K_1 and K_2 by $\lambda_1=\lambda_1(\xi)$ and $\lambda_2=\lambda_2(\xi)$ respectively, where λ means the spherical radius of curvature of K. Then the specific slidings of F_1 and F_2 are given by Equation (12) in the report (VI) as follows respectively:

$$(5)\quad \sigma_1=\sigma_1(\xi)=\frac{\dfrac{1}{\tan\lambda_1(\xi)}-\dfrac{1}{\tan\lambda_2(\xi)}}{\dfrac{1}{\tan\lambda_r}-\dfrac{1}{\tan\lambda_1(\xi)}},\quad \sigma_2=\sigma_2(\xi)=\frac{\dfrac{1}{\tan\lambda_2(\xi)}-\dfrac{1}{\tan\lambda_1(\xi)}}{\dfrac{1}{\tan\lambda_r}-\dfrac{1}{\tan\lambda_2(\xi)}}.$$

From (5) we can derive the following results:

i. *The values of σ_1 and σ_2 are independent of the position of the drawing point C.*

ii. *When the pitch curves K_1 and K_2 are both circles, then the specific slidings σ_1 and σ_2 are both constants. Conversely, when both K_1 and K_2 are circles and both the specific slidings σ_1, σ_2 of F_1 and F_2 are constants, then the rolling curve K_r for F_1 and F_2 must be necessarily a circle.*[1]

§ 7. Spherical circular profile curves.

Take a small circle K with spherical radius λ as a pitch curve and settle at K a circular arc F with center M and spherical radius ρ. When the point M exists inside of K, then any arc of the small circle F may be adopted as a profile curve. When M exists outside of K, then any part of the arc between the two tangent great circles drawn from M to K may be adopted as a profile curve. When M exists on the perimeter of K, then any arc of F can not be adopted as a profile curve making one-point contact motion.

Give orientation to the pitch circle K such that the spherical radius λ is positive and take as origin one of two points at which the great circle connecting the center O of K with the center M of F intersects the perimeter of K, the nearer one P_0 to M.

Now we may consider the curve F as a parallel profile curve with spherical distance ρ from the point M. In this case the direction of F and accordingly the sign of ρ are self-determined. From Equation (1) of M we have the equation of F by Equation (3) in the report (V) as follows:

when $\delta>0$

$$(6)_1\quad \varphi=f(\xi)=\pm\rho+\cos^{-1}\left\{\sin\lambda\sin(\lambda-\delta)\cos\left[\frac{\xi}{\sin\lambda}\right]+\cos\lambda\cos(\lambda-\delta)\right\},$$

when $\delta<0$

$$(6)_2\quad \varphi=f(\xi)=\begin{cases}-\rho-\cos^{-1}\left\{\sin\lambda\sin(\lambda-\delta)\cos\left[\dfrac{\xi}{\sin\lambda}\right]+\cos\lambda\cos(\lambda-\delta)\right\},\\ \quad\text{where}\quad |\xi|\leq\sin\lambda\cos^{-1}\left[\dfrac{\tan\lambda}{\tan(\lambda-\delta)}\right],\end{cases}$$

1) T. Kubota, Geometry of Gears (Japanese) (1917) p. 160.

$$\begin{cases} -\rho+\cos^{-1}\left\{\sin\lambda\sin(\lambda-\delta)\cos\left[\frac{\xi}{\sin\lambda}\right]+\cos\lambda\cos(\lambda-\delta)\right\}, \\ \quad \text{where} \quad |\xi|\geqq\sin\lambda\cos^{-1}\left[\frac{\tan\lambda}{\tan(\lambda-\delta)}\right]. \end{cases}$$

The path of contact of F is a conchoid curve with spherical distance ρ from the circular arc with center O and spherical radius $\lambda-\delta$, and then its equation is given by Equation (7) in the report (V) as follows:

when $\delta>0$

$$(7)_1 \quad \varphi=g(\theta)=\pm\rho+2\tan^{-1}\left\{\frac{\sin\lambda\sin\theta\pm\sqrt{\sin^2\lambda\sin^2\theta-\sin(2\lambda-\delta)\sin\delta}}{\cos(\lambda-\delta)+\cos\lambda}\right\},$$

when $\delta<0$

$$(7)_2 \quad \varphi=g(\theta)=\begin{cases} -\rho+2\tan^{-1}\left\{\frac{\sin\lambda\sin\theta+\sqrt{\sin^2\lambda\sin^2\theta-\sin(2\lambda-\delta)\sin\delta}}{\cos(\lambda-\delta)+\cos\lambda}\right\}, \\ \quad \text{where} \quad \theta\geqq 0, \\ -\rho+2\tan^{-1}\left\{\frac{\sin\lambda\sin\theta-\sqrt{\sin^3\lambda\sin^2\theta-\sin(2\lambda-\delta)\sin\delta}}{\cos(\lambda-\delta)+\cos\lambda}\right\}, \\ \quad \text{where} \quad \theta\leqq 0. \end{cases}$$

Now we put $\lambda-\delta+\rho=\beta$. β represents the spherical distance of the point O and the small circle F. Then the first equation of $(6)_2$ becomes

$$(8) \quad \varphi=f(\xi)=-\rho-\cos^{-1}\left\{\sin\lambda\sin(\beta-\rho)\cos\left[\frac{\xi}{\sin\lambda}\right]+\cos\lambda\cos(\beta-\rho)\right\}.$$

When, in this case, ρ tends to $-\frac{\pi}{2}$, then the curve F becomes a part of a great circle and its equation is given by

$$(9) \quad \varphi=f(\xi): \quad \sin\varphi=\sin\lambda\cos\beta\cos\left[\frac{\xi}{\sin\lambda}\right]-\cos\lambda\sin\beta,$$

where $|\xi|\leqq\sin\lambda\cos^{-1}[-\tan\lambda\tan\beta]$,

and the curve of contact of F is given by

$$(10) \quad \varphi=g(\theta): \quad \tan\frac{\varphi}{2}=\frac{-\cos\lambda+\sqrt{\cos^2\beta-\sin^2\lambda\cos^2\theta}}{\sin\lambda|\sin\theta|+\sin\beta}$$

Now if we transform the origin from the point P_0 into one of the points of intersection of the great circle and the small circle K, we have, substituting $\xi+\sin\lambda\cos^{-1}\left[\frac{\tan\beta}{\tan\lambda}\right]$ or $\xi-\sin\lambda\cos^{-1}\left[\frac{\tan\beta}{\tan\lambda}\right]$ in place of ξ in (9),

$$(11) \quad \varphi=f(\xi): \quad \sin\varphi=\cos\lambda\sin\beta\left\{\cos\left[\frac{\xi}{\sin\lambda}\right]-1\right\}+\sin\theta_0\sin\lambda\sin\left[\frac{\xi}{\sin\lambda}\right],$$

where θ_0 denotes the angle of intersection of F and K.

§ 3. Octoid profile curves.

In Equation (11), if we make moreover λ tend to $\frac{\pi}{2}$, we have

(12) $$\varphi=f(\xi): \quad \sin\varphi=\sin\theta_0\sin\xi$$

If we adopt a great circle K as a pitch curve and an arc F of a great circle which intersects K at angle θ_0 as a profile curve, the equation of F is given by (12).

When we take a small circle K_1 as a pitch curve corresponding to K, the profile curve F_1 corresponding to F is called an octoid profile curve. If we take again a small circle K_2 as another pitch curve corresponding to K, we obtain again an octoid profile curve F_2. When we assort K_1 and K_2 as a pair of pitch curves, the octoid curves F_1 and F_2 become a pair of profile curves. The equation of F_1 or F_2 is given of course by (12), and their curve of contact Γ is given by

(13) $$\varphi=g(\theta): \quad \cos\varphi=\frac{\cos\theta_0}{|\sin\theta|}, \quad \mathrm{sgn}(\varphi)=\mathrm{sgn}(\theta),$$

by putting $\lambda=\frac{\pi}{2}$ into Equatioh (10) or by eliminating ξ from (12) using the relation

(14) $$\frac{d\varphi}{d\xi}=-\mathrm{sgn}(\theta)\cos\theta.$$

Next, the equation of the rolling curve K_r for the octoid profile curves F_1 and F_2 is given by Equation (5) in the report (V) as follows:

(15) $$\lambda_r=\lambda_r(\xi): \quad \frac{1}{\tan\lambda_r(\xi)}=\frac{\cot\theta_0}{\sin\xi\,(1-\sin^2\theta_0\sin^2\xi)}.$$

And then the specific slidings of F_1 and F_2 are

(16)
$$\sigma_1=\sigma_1(\xi)=\frac{\dfrac{1}{\tan\lambda_1(\xi)}-\dfrac{1}{\tan\lambda_2(\xi)}}{\dfrac{\cot\theta_0}{\sin\xi(1-\sin^2\theta_0\sin^2\xi)}-\dfrac{1}{\tan\lambda_1(\xi)}}=\frac{\dfrac{1}{\tan\lambda_1}-\dfrac{1}{\tan\lambda_2}}{\dfrac{\cos\theta_0}{\sin\varphi\cos^2\varphi}-\dfrac{1}{\tan\lambda_1}}.$$
$$\sigma_2=\sigma_2(\xi)=\frac{\dfrac{1}{\tan\lambda_2(\xi)}-\dfrac{1}{\tan\lambda_1(\xi)}}{\dfrac{\cot\theta_0}{\sin\xi(1-\sin^2\theta_0\sin^2\xi)}-\dfrac{1}{\tan\lambda_2(\xi)}}=\frac{\dfrac{1}{\tan\lambda_2}-\dfrac{1}{\tan\lambda_1}}{\dfrac{\cos\theta_0}{\sin\varphi\cos^2\varphi}-\dfrac{1}{\tan\lambda_2}}.$$

§4. Spherical involute profile curves.

We take a great circle K and a small circle K_1 which touchs K at a point P_0 on K as a pair of pitch curves and give K_1 (and accordingly K) a direction such that the spherical radius λ_1 is positive. In this case, if we adopt an arc of a great circle passing the point P_0 as a profile curve settled at K, the profile curve F_1 of K_1 corresponding to F is an octoid curve and its patch of contact Γ is given by Equation (14) as we discussed in the preceding paragraph.

Now we adopt an arc of great circle passing the point P_0 as a path of contact Γ and consider the profile curves which correspond to this Γ and have the same pitch curves K and K_1. Denote by θ^* the angle of intersection of Γ and K. We may suppose the angle θ^* to be positive. When we draw a great circle intersecting K_1 with angle θ^* passing each point on K_1, then we obtain a small circle with spherical radius $\sin^{-1}[\sin\lambda_1|\cos\theta^*|]$ and concentric with K_1 as an envelope of the family of those great circles. Accordingly, a spherical involute F_1 which is drawn out from this small circle and just passes through the point P_0 becomes a profile curve of K_1 corresponding to Γ.

The path of contact Γ is represented by

$$\text{(17)} \qquad \varphi = g(\theta): \qquad \theta = \theta^* + \{\operatorname{sgn}(\theta_0)\operatorname{sgn}(\xi) - 1\}\frac{\pi}{2},$$

where

$$\theta_0 = \theta^* - \frac{\pi}{2}.$$

The equation of the involute profile curve F_1 and accordingly of the profile curve F of K corresponding to F_1 is derived from (17) and (14):

$$\text{(18)} \qquad \varphi = f(\xi) = \xi \sin\theta_0.$$

The rolling curve K_r for the involute profile curve F_1 is

$$\text{(19)} \qquad \lambda_r = \lambda_r(\xi): \qquad \frac{1}{\tan\lambda_r} = \frac{\cos\theta_0}{\tan\varphi}.$$

If we take a small circle K_2 as another pitch curve corresponding to the pitch curve K, we obtain again an involute profile curve F_2 drawn out from a small circle with spherical radius $\sin^{-1}[\sin\lambda_2|\cos\theta^*|]$ and concentric with K_2. When we adopt these two small circles K_1 and K_2 as a pair of pitch curves, the spherical involutes F_1 and F_2 consist of a pair of profile curves.

There happens a case in which we must adopt as a part of profile curves that of the another more one spherical involute drawn out from the starting point on the base circle of F_1 or F_2. The case is quite the same as that which was mentioned concerning the plane involute profile curves in the report (IV), § 3.

Moreover the specific slidings of F_1 and F_2 are given by

$$\text{(20)} \qquad \sigma_1 = \frac{\dfrac{1}{\tan\lambda_1} - \dfrac{1}{\tan\lambda_2}}{\dfrac{\cos\theta_0}{\tan\varphi} - \dfrac{1}{\tan\lambda_1}}, \quad \sigma_2 = \frac{\dfrac{1}{\tan\lambda_2} - \dfrac{1}{\tan\lambda_1}}{\dfrac{\cos\theta_0}{\tan\varphi} - \dfrac{1}{\tan\lambda_2}}.$$

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