35. Notes on Fourier Analysis (XXXII). On the Summability (C, 1) of the Fourier Series.

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1. Let $f(x)$ be an L-integrable and periodic function with period 2π. Concerning the summability $(C, 1)$ of the Fourier series of $f(x)$, Hahn[1] has proved the following theorem.

Theorem A. *If*

$$(1) \qquad \int_0^t \varphi(x, u)\, du = o(t) \quad (t \to o),$$

where $$\varphi(x, u) = \frac{1}{2}\{f x + u) + f(x-u) - 2f(x)\},$$

then the Fourier series of $f(x)$ *is summable* $(C, 1+\delta)$ $(\delta > 0)$, *but not necessary summable* $(C, 1)$.

Prasad[2] has replaced (1) by the condition that

$$(2) \qquad \int_0^t \varphi(x, u)\, u^{-1}\, du$$

exists by the Cauchy's sense.

On the other hand Hsiang[3] has recently proved the following theorem:

Theorem B. *If for any* $\eta > 0$,

$$(3) \qquad \int_0^t \varphi(x, u)\, u^{-(1+\eta)}\, du$$

exists by the Cauchy's sense, then the Fourier series of $f(x)$ *is summable* $(C, 1)$ *but not necessary summable* $(c, (1+\eta)^{-1} - \varepsilon)$ $\varepsilon > 0$.

Our object of this paper is to prove the following theorems.

Theorem 1. *If for any* $\delta > 0$,

$$(4) \qquad \int_0^t \varphi(x, u)(\log 1/u)^{1+\delta}\, u^{-1}\, du$$

exists by the Cauchy's sense, then the Fourier series of $f(x)$ *is summable* $(C, 1)$ *at the point* x.

Theorem 2. *If for any* $s \geq 0$,

$$(5) \qquad \int_0^t \varphi(x, u)(\log 1/u)^s\, u^{-1}\, du$$

exists by the Cauchy's sense, then the Fourier series of $f(x)$ *is summable* $(R, \log, 1)$ *at the point* x.

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1) Hahn: Jour. Deuts. Math. Ver., **25** (1916).
2) Prasad: Math. Zeits., **40** (1935).
3) Hsiang: Duke Math. Jour., **13** (1946).

Theorem 3. *For any* $0 \leq s < 1$ *there exists a function* $f(x)$ *satisfing the condition* (5) *but the Fourier series of* $f(x)$ *is not summable* $(C, 1)$ *at the point* x.

2. **Lemma.** *If for any* $s > 0$ *the integral* (5) *exists by the Cauchy's sense, then*

$$\int_0^t \varphi(u)\, du = o\,(t\,(\log 1/t)^{-s}),$$

and

$$\int_0^t \varphi(u)\, u^{-1}\, du = o\,((\log 1/t)^{-s}).$$

Proof. Let us put

$$\Phi_\varepsilon(t) = \int_\varepsilon^t \varphi(u)\,(\log 1/u)^s\, u^{-1}\, du$$

for any ε. Then for any $\eta > 0$, there exist $t_1 = t_1(\eta)$ such that $|\Phi_\varepsilon(t)| < \eta$ for $0 < \varepsilon \leq t \leq t_1$.

$$\int_\varepsilon^t \varphi(u)\, du = \int_\varepsilon^t \varphi(u) \frac{1}{u} (\log 1/u)^s \frac{u}{(\log 1/u)^s}\, du$$

$$= \Phi_\varepsilon(t)\, t\,(\log 1/t)^{-s} - \int_\varepsilon^t \Phi_\varepsilon(u)\,\{(\log 1/u)^{-s} + s\,(\log 1/u)^{-(s+1)}\}\, du.$$

Consequently if $\varepsilon \leq t \leq t_1$, then

$$\left|\int_\varepsilon^t \varphi(u)\, du\right| \leq \eta\, t\,(\log 1/t)^{-s} + \int_\varepsilon^t \eta\,\{(\log 1/u)^{-s} + s\,(\log 1/u)^{-(s+1)})\}\, du$$

$$\leq \eta\, t\,(\log 1/t)^{-s} + \eta\, t\,\{(\log 1/t)^{-s} + s\,(\log 1/t)^{-(s+1)}\} \leq \eta\, t\,(\log 1/t)^{-s},$$

Thus the first half of Lemma is proved. Remaining part is proved by the similar way.

Let $\sigma_n(x)$ be the $(C, 1)$-mean of the Fourier series of $f(x)$ at the point x. Then we have

$$(6) \qquad \sigma_n(x) - f(x) = \frac{1}{2\pi n} \int_0^\pi \varphi(x, t) \left(\frac{\sin (n+1)\, t/2}{\sin t/2}\right)^2 dt$$

$$= \frac{1}{2\pi n} \int_0^\pi \varphi(x, t) \left(\frac{\sin nt}{t}\right)^2 dt + o(1)$$

$$= \frac{1}{2\pi} \int_0^\pi \varphi_1(t) \sin 2nt/t^2\, dt + \frac{1}{\pi n} \int_0^\pi \varphi_1(t) \sin^2 nt/t^3\, dt + o(1),$$

where

$$\varphi_1(t) = \int_0^t \varphi(x, u)\, du.$$

From Lemma and (4),

$$\varphi_1(t)/t^2 = o\,((\log 1/t)^{1+\delta}/t).$$

Hence by the Riemann Lebesgue's theorem the first term of the right hand side of (6) is $o(1)$. On the other hand by the same reason

$$\varphi_1(t)/t = o\,((\log 1/t)^{1+\delta}) = o(1) \quad (t \to 0).$$

Consequently, by the Fejér's theorem, the second term of the right hand side of (6) is $o(1)$.

Thus Theorem 1 is proved.

For the proof of Theorem 2 it is sufficient to prove the case $s=0$. Let $R_n(x)$ be the $(R, \log, 1)$-mean of the Fourier series of $f(x)$ at the point x.[4)]

$$R_n(x)-f(x)=\frac{1}{\pi}\frac{n}{\log n}\int_0^\pi \varphi(t)\,L_1(nt)+o(1).$$

Now

$$\frac{n}{\log n}\int_\varepsilon^\pi \varphi(t)\,L_1(nt)\,dt=\frac{n}{\log n}\left\{\Big[\Phi_\varepsilon(t)\,tL_1(nt)\Big]_\varepsilon^\pi-\int_\varepsilon^\pi \Phi_\varepsilon(t)\,L_0(nt)\,dt\right\}$$

$$=\frac{n}{\log n}\left\{\Phi_\varepsilon(\pi)\,\pi L_1(n\pi)-\int_\varepsilon^\pi \Phi_\varepsilon(t)\sin nt/nt\,dt\right\}\equiv P-Q,$$

say, where

$$\Phi_\varepsilon(t)=\int_\varepsilon^t \varphi(u)\,u^{-1}\,du.$$

We have

$$P=O\left(\frac{n}{\log n}\right)O(1/n\pi)=O(1/\log n)=o(1).$$

Secondly

$$Q=\frac{n}{\log n}\left\{\int_\varepsilon^{1/n}+\int_{1/n}^{t_1}+\int_{t_1}^\pi\right\}\Phi_\varepsilon(t)\sin nt/nt\,dt\equiv Q_1+Q_2+Q_3,$$

say. For $\varepsilon\leqq t\leqq t_1$, we have

$$\left|\Phi_\varepsilon(t)\right|=\left|\int_\varepsilon^\pi \varphi(u)\,u^{-1}\,du\right|<\eta$$

$$|Q_1|\leqq\frac{n}{\log n}\int_\varepsilon^{1/n}\eta\,nt/nt\,dt\leqq\eta/\log n=o(1).$$

$$|Q_2|\leqq\frac{n}{\log n}\int_{1/n}^{t_1}\eta/nt\,dt\leqq\eta/\log nt\,(\log nt)=\eta+o(1).$$

$$|Q_3|\leqq\frac{n}{\log n}\int_{t_1}^\pi O(1)/nt\,dt=O(1/\log n)=o(1).$$

That is,

$$\frac{n}{\log n}\int_\varepsilon^\pi \varphi(t)\,L_1(nt)\,dt=o(1)$$

uniformly in ε. Thus the theorem is proved.

3. Let $\{p_k\}$ be an increasing sequence of positive integers and $\{C_k\}$ be a positive sequence, especially $c_1=0$. We define the functions $F(t)$ and $\varphi_1(t)$ in the following manner.

If t is a point of the interval $J_k\equiv(\pi/p_k,\ \pi/p_{k-1})$, let

$$F(t)=c_k\sin p_k t$$

and

$$\varphi_1(t)=F(t)\,t\,(\log 1/t)^{-s},$$

where

$$0\leqq s<1.$$

4) Wang: Tôhoku Math. Jour., 40 (1935).

1° *The condition for which* $\varphi_1'(t) \in L(0, \pi)$.

$$\int_0^\pi |\varphi_1'(t)|\, dt \leqq \sum_{k=1}^\infty \int_{J_k} | c_k p_k \cos p_k t\, t (\log 1/t)^{-s}$$
$$+ c_k \sin p_k t \{(\log 1/t)^{-s} + s(\log 1/t)^{-(s+1)}\} |\, dt$$
$$\leqq \sum_{k=1}^\infty c_k p_k \int_{\pi/p_k}^{\pi/p_{k-1}} t(\log 1/t)^{-s}\, dt + \sum_{k=1}^\infty c_k \int_{\pi/p_k}^{\pi/p_{k-1}} \{(\log 1/t)^{-s} + s(\log 1/t)^{-(s+1)}\}\, dt$$
$$\leqq \sum_{k=1}^\infty c_k p_k (\log p_{k-1})^{-s} p_{k-1}^{-2} + \sum_{k=1}^\infty c_k \{(\log p_{k-1})^{-s} + s(\log p_{k-1})^{-(s+1)}\}/p_{k-1}$$
$$\leqq \sum_{k=1}^\infty c_k p_k p_{k-1}^{-2} (\log p_{k-1})^{-s}. \qquad (7)$$

Consequently if the series (7) is convergent then $\varphi_1'(t)$ is integrable. Hence we define $\varphi(t)$ by

$\varphi(t) \equiv \varphi_1(t) = c_k p_k \cos p_k t \cdot t (\log 1/t)^{-3} + c_k \sin p_k t \{(\log 1/t)^{-s} + s(\log 1/t)^{-(s+1)}\}$

for $t \leftarrow J_k$, $\varphi(-t) = \varphi(t)$ and $\varphi(2\pi + t) = \varphi(t)$ for any t. Since $\varphi(t)$ is an integrable and even periodic function with period 2π, we can write

$$\varphi(t) \sim \sum_0^\infty a_n \cos nt.$$

Especially $\qquad a_0 = 0, \quad$ for $\quad \varphi_1(\pi) = 0$.

We consider the summability of the Fourier series of $\varphi(t)$ at $t = 0$, and we prove that it is not summable $(C, 1)$.

2° *The condition for which* (5) *is satisfied.*

$$\int_\varepsilon^t \varphi(t) (\log 1/t)^s/t\, dt = [\varphi_1(t)(\log 1/t)^s/t - \varphi_1(\varepsilon)(\log 1/\varepsilon)^s/\varepsilon]$$
$$+ \int_\varepsilon^t \varphi_1(t) \{t^{-2}(\log 1/t)^s + st^{-2}(\log 1/t)^{s-1}\}\, dt,$$

where if $\qquad \varepsilon \in J_k$,

$$\varphi_1(\varepsilon)(\log 1/\varepsilon)^s/\varepsilon = F(\varepsilon) = c_k \sin p_k \varepsilon.$$

Hence the function $\varphi(t)$ satisfies the condition (5) if there exists

$$\lim_{\varepsilon \to 0} \int_\varepsilon^t \varphi_1(t) t^{-2} (\log 1/t)^s\, dt,$$

and $\qquad c_k = o(1)$.

For any $t \in J_k$

$$\left| \int_\varepsilon^t \varphi_1(u)(\log 1/u)^s u^{-2}\, dt \right| \leqq \sum_{i=k}^\infty \left| \int_{\pi/p_i}^{\pi/p_{i-1}} c_i \sin p_i u/u\, du \right|$$
$$\leqq \frac{1}{\pi} \sum_{i=k}^\infty c_i p_i/p_i \leqq \frac{1}{\pi} \sum_{i=1}^\infty c_i.$$

Consequently if $\sum c_i < \infty$, then $\varphi(t)$ satified the condition (5).

3° *The condition for which the Fourier series is not summable* $(C, 1)$ *at* $t = 0$.

$$2\pi\,(\sigma_{p_k}(0)-f(0)) = \int_0^{\pi}\varphi_1(t)\,t^{-2}\sin p_k t\,dt + o(1)$$

$$= \left(\int_0^{\pi/p_k} + \int_{\pi/p_k}^{\pi/p_{k-1}} + \int_{\pi/p_{k-1}}^{\pi}\right) + o(1) \equiv S_1 + S_2 + S_3 + o(1),$$

say.

$$S_1 = \sum_{i=k+1}^{\infty}\int_{\pi/p_i}^{\pi/p_{i-1}} c_i \sin p_i t\,(\log 1/t)^{-s}/t\,dt$$

$$= \sum_{i=k+1}^{\infty}\frac{c_i}{2}\int_{\pi/p}^{\pi/p_{i-1}}\{\cos(p_i-p_k)t+\cos(p_i+p_k)t\}(\log 1/t)^{-s}/t\,dt.$$

$$S_1 \leqq \sum_{i=k+1}^{\infty}\frac{c_i}{2}\,p_i(\log p_i)^{-s}\left(\frac{1}{p_i-p_k}+\frac{1}{p_i+p_k}\right)$$

$$= \sum_{i=k+1}^{\infty}\frac{c_i p_i}{2(\log p_i)^s}\cdot\frac{p_i}{p_i^2-p_k^2} = A\sum_{i=k+1}^{\infty}c_i(\log p_i)^{-s},$$

$$S_2 = \frac{c_k}{2}\int_{\pi/p_k}^{\pi/p_{k-1}}\frac{1-\cos 2p_k t}{t(\log 1/t)^s}\,dt$$

$$= \frac{c_k}{2}\left[(\log p_{k-1})^{1-s} - (\log p_k)^{1-s}\right] + c_k(\log p_k)^{-s},$$

$$|S_3| \leqq A\sum_{i=1}^{k-1}\frac{c_i}{2}(\log p_i)^{-s}.$$

Hence if $S_1 = o(1)$, $S_2 \to \infty$, and $S_3 = O(1)$ for $k\to\infty$, the Fourier series of $\varphi(t)$ is not summable $(C, 1)$ at $t=0$. Or

$$\sum_{i=1}^{\infty} c_i(\log p_i)^{-s} < \infty,$$

$$c_k[(\log p_{k-1})^{1-s} - (\log p_k)^{1-s}] \to \infty \ (k\to\infty).$$

Let $p_k = p_1^{2^{k-1}} = 2^{2^k}$ and $c_k = 2^{-\varepsilon k(1-s)}$, $0<\varepsilon<1$, then all conditions 1°–3° are satisfied and then Theoren 3 is proved.