48. On the Notion of Measurability.

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Our main purpose of this paper is to give another definition of "Measurability" with respect to Carathéodory's outer measure than that which is given by Carathéodory himself.

Let X be a metric space. In this paper we shall consider Carathéodory's outer measure[1] μ defined for all sub-sets of X which satisfies the following condition:

$$(1) \qquad \mu(A) < +\infty \qquad \text{for every bounded set } A,$$

and call it *conditionally finite outer measure*. The class of sets that are measurable (with respect to μ) in the sense of Carathéodory is denoted by $\mathfrak{C}(\mu)$.

Consider a class $\mathfrak{K}$ of sets which consists of the elements of $\mathfrak{C}(\mu)$ and in which there exists a sequence $\{K_n\}$ of bounded sub-sets such that $\bigcup_{n=1}^{\infty} K_n = X$ and $K_n \subseteq K_{n+1}$ for every n. Let $\mathfrak{R}(\mathfrak{K}, \{K_n\}, \mu)$ be the class of sets A satisfying $\mu(A_n) = \mu(K_n) - \mu(K_n \frown CA_n)$[2] for every n, where A_n is the common part A and K_n. For instance, for $\mathfrak{K}$ and $\{K_n\}$ we can take the class of circles $S_n(p)$ of n-radius with a fixed point p. We can see at once that the class $\mathfrak{R}(\mathfrak{K}, \{K_n\}, \mu)$ is independent on $\mathfrak{K}$ and $\{K_n\}$ and therefore we denote it by $\mathfrak{R}(\mu)$ and call *μ-modular set* the element of $\mathfrak{R}(\mu)$. When $\mathfrak{M}$ is a completely additive class and μ is completely additive on $\mathfrak{M}$, we say that $\mathfrak{M}$ is *μ-completely additive class*. Given classes of sets $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $[\mathfrak{M}, \mathfrak{N}]$ the smallest completely additive class containing $\mathfrak{M}$ and $\mathfrak{N}$.

Definition. Let μ be conditionally finite outer measure given in a metric space X. Let m be an arbitrary regular[3] outer measure satisfying the property such that $m(K_n) = \mu(K_n)$ for every K_n and that $m(A) \geq \mu(A)$ for the other sets A. Then, m is termed *dominant measure of* μ and any set A of $\mathfrak{R}(m)$ is said to be *m-dominant measurable set of* μ. Particularly, consider the set function $\mu_0(A) = \inf m_\alpha(A)$, where the infimum is taken over all dominant measure m_α of μ. In this case, if μ_0 is a regular outer measure and therefore a dominant measure of μ, the outer measure μ is called *relatively regular* and any μ_0-dominant measurable set of μ is called *μ-measurable set.*

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1) C. Carathéodory: Vorlesungen über reelle Funktion (1927), § 235.

2) CA is complement of A.

3) C. Carathéodory: Loc. cit. § 253.

Let us begin with the following three lemmas.

Lemma 1. *Given a sequence of sets E_n such that E_n is measurable with respect to an outer measure μ and that $E_n \subseteq E_{n+1}$ $(n = 1, 2, \ldots)$, we have for an arbitrary set A*

$$\lim_{n\to\infty} \mu(A \frown E_n) = \mu(A \frown (\bigcup_{n=1}^{\infty} E_n)) .$$

Since this proof is easy, we shall omit here.

Lemma. 2. *In order that the class $\mathfrak{R}(\mu)$ of μ-modular sets be μ-completely additive class, where μ is a conditionally finite outer measure defined in X, it is necessary and sufficient that the following equality should hold*

$$(2) \qquad \mu(A \frown B) + \mu(A \frown \mathrm{C}B) = \mu(A) \quad \textit{whenever } A,\ B \in \mathfrak{R}(\mu) .$$

Proof. It is evident that the condition (2) is necessary. We shall prove the sufficiency of the condition.

i) Evidently, if $A \in \mathfrak{R}(\mu)$, then $\mathrm{C}(A) \in \mathfrak{R}(\mu)$.

ii) If $A,\ B \in \mathfrak{R}(\mu)$, then $A \frown B$, $A \smile B \in \mathfrak{R}(\mu)$. By the definition, it is enough to show the case when A and B are sub-sets of some K_n. Using $A,\ B \in \mathfrak{R}(\mu)$;

$$\begin{aligned}\mu(K_n) &= \mu(K_n \frown \mathrm{C}(A \frown B)) = \mu(A) + \mu(K_n \frown \mathrm{C}A) - \mu(K_n \frown \mathrm{C}(A \frown B))\\ &= \mu(A \frown B) + \mu(A \frown \mathrm{C}B) + \mu(K_n \frown \mathrm{C}A) - \mu(K_n \frown \mathrm{C}(A \frown B))\\ &\geqq \mu(A \frown B) + \mu(A \frown \mathrm{C}B) + \mu(K_n \frown \mathrm{C}A) - \mu(A \frown \mathrm{C}B) - \mu(K_n \frown \mathrm{C}A)\\ &= \mu(A \frown B).\end{aligned}$$

On the other hand, $\mu(K_n) - \mu(K_n \frown \mathrm{C}(A \frown B)) \leqq \mu(A \frown B)$. Therefore $\mu(A \frown B) = \mu(K_n) - \mu(K_n \frown \mathrm{C}(A \frown B))$, whence $A \frown B \in \mathfrak{R}(\mu)$. From i) and $A \frown B \in \mathfrak{R}(\mu)$, $A \smile B \in \mathfrak{R}(\mu)$ follows.

iii) If $A,\ B \in \mathfrak{R}(\mu)$ and $A \frown B = 0$, then $\mu(A \smile B) = \mu(A) + \mu(B)$. It is evident from $A \smile B \in \mathfrak{R}(\mu)$ and (2).

iv) If $A_i \in \mathfrak{R}(\mu)$ and $A_i \frown A_j = 0$ $(i \neq j)$, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. If we put $A = \bigcup_{i=1}^{\infty} A_i$, since $A \supseteqq \bigcup_{i=1}^{n} A_i$, $\mu(A) \geqq \mu(\bigcup_{i=1}^{n} A_i)$ and therefore $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i) \leqq \mu(A)$. On the other hand, since μ is the outer measure, $\sum_{i=1}^{\infty} \mu(A_i) \geqq \mu(A)$ and therefore $\sum_{i=1}^{\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$.

v) If $A_i \in \mathfrak{R}(\mu)$ $(i = 1, 2, \ldots)$, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{R}(\mu)$. It is enough to show the case when $A_i \frown A_j = 0$ $(i \neq j)$ and $A_i \subseteq K_n$ $(i = 1, 2, \ldots)$ for some K_n. Now by iv),

$$\begin{aligned}\mu(K_n) - \mu(\bigcup_{i=1}^{\infty} A_i) &= \mu(K_n) - \sum_{i=1}^{\infty} \mu(A_i) = \lim_{j\to\infty} (\mu(K_n) - \sum_{i=1}^{j} \mu(A_i))\\ &= \lim_{j\to\infty} (\mu(K_n) - \mu(\bigcup_{i=1}^{j} A_i)) = \lim_{j\to\infty} \mu(K_n \frown \mathrm{C}(\bigcup_{i=1}^{j} A_i)) \geqq \mu(K_n \frown \mathrm{C}(\bigcup_{i=1}^{\infty} A_i)),\end{aligned}$$

whence $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{R}(\mu)$ follows.

Lemma 3. *Let μ be a conditionally finite outer measure defined in X and $\mathfrak{M}$ the class of sets which belongs to at least one of classes $\mathfrak{M}_\beta$ which are μ-completely additive class of sub-sets of X and contains $\mathfrak{C}(\mu)$. Then $\mathfrak{M}$ coincides with the class $\mathfrak{R}(\mu)$ of μ-modular sets.*

Proof. It is easy to see $\mathfrak{M} \subseteq \mathfrak{R}(\mu)$. Let us prove $\mathfrak{M} \supseteq \mathfrak{R}(\mu)$. Firstly, let us show that $[\mathfrak{C}(\mu), A]$ for any set A which belongs to $\mathfrak{R}(\mu)$, is μ-completely additive class. We see easily that, if $M \in [\mathfrak{C}(\mu), A]$, M is given by the form such that

$$(3) \qquad M = (B \frown A) \smile (B' \frown CA), \quad \text{where} \quad B,\ B' \in \mathfrak{C}(\mu)\,.$$

And moreover the following equality holds

$$(4) \qquad \mu(M) = \mu(B \frown A) + \mu(B' \frown CA)\,.$$

In order to show the equation, by Lemma 1, it is enough to prove the case when A, B and B' are sub-sets of some K_n. Now, for any set $E \in \mathfrak{C}(\mu)$ such that $E \subseteq K_n$,

$$\begin{aligned}
\mu(K_n) &= \mu(E) + \mu(K_n \frown CE) \\
&\leqq \mu(E \frown A) + \mu(E \frown CA) + \mu(K_n \frown CE \frown A) + \mu(K_n \frown CE \frown CA) \\
&= \mu(A \frown E) + \mu(A \frown CE) + \mu(K_n \frown CA \frown E) + \mu(K_n \frown CA \frown CE) \\
&= \mu(A) + \mu(K_n \frown CA) = \mu(K_n)\,.
\end{aligned}$$

Thus, $\mu(E) + \mu(K_n \frown CE) = \mu(E \frown A) + \mu(E \frown CA) + \mu(K_n \frown CE \frown A) + \mu(K_n \frown CE \frown CA)$. Therefore,

$$(5) \qquad \mu(E) = \mu(E \frown A) + \mu(E \frown CA)\,.$$

Using (5 ,

$$\begin{aligned}
\mu(B \smile B') &= \mu((B \smile B') \frown A) + \mu((B \smile B') \frown CA) \\
&= \mu(B \frown A) + \mu(B' \frown CB \frown A) + \mu(B' \frown CA) + \mu(B \frown CB' \frown CA) \\
&\geqq \mu((B \frown A) \smile (B' \frown CA)) + \mu((B' \frown CB \frown A) \smile (B \frown CB' \frown CA) \\
&\geqq \mu(B \smile B')\,.
\end{aligned}$$

whence $\mu((B \frown A) \smile (B' \frown CA)) = \mu(B \frown A) + \mu(B' \frown CA)$.

Let $M_i \in [\mathfrak{C}(\mu), A]$, $M_i \frown M_j = 0$ $(i \neq j)$, then they are given by $M_i = (B_i \frown A) \smile (B'_i \frown CA)$, $B_i \frown B_j = 0$ $(i \neq j)$ and $B'_i \frown B'_j = 0$ $(i \neq j)$. Using (4), (5) and that $\mu((\bigcup_{i=1}^{\infty} B_i) \frown A) = \sum_{i=1}^{\infty} \mu(B_i \frown A)$ by Lemma 1, it follows that

$$\begin{aligned}
\mu(\bigcup_{i=1}^{\infty} M_i) &= \mu(\bigcup_{i=1}^{\infty}((B_i \frown A) \smile (B'_i \frown CA))) \\
&= \mu((\bigcup_{i=1}^{\infty} B_i) \frown A) \smile (\bigcup_{i=1}^{\infty} B'_i) \frown CA) \\
&= \mu((\bigcup_{i=1}^{\infty} B_i) \frown A) + \mu((\bigcup_{i=1}^{\infty} B'_i) \frown CA) \\
&= \sum_{i=1}^{\infty} \mu(B_i \frown A) + \sum_{i=1}^{\infty} \mu(B'_i \frown CA) \\
&= \sum_{i=1}^{\infty} \mu((B_i \frown A) \smile (B'_i \frown CA)) \\
&= \sum \mu(M_i)\,.
\end{aligned}$$

Consequently, $[\mathfrak{C}(\mu), A]$ is μ-completely additive class. The class of sets belonging to $[\mathfrak{C}(\mu), A]$, where A is taken over all elements of $\mathfrak{R}(\mu)$, contains $\mathfrak{R}(\mu)$ and therefore $\mathfrak{M} \supseteq \mathfrak{R}(\mu)$.

Corollary 1. *In order that the class $\mathfrak{R}(\mu)$ of μ-modular sets of a conditionally finite outer measure μ defined in X be μ-completely additive class, it is necessary and sufficient that $\mathfrak{M}$ be μ-completely additive class.*

Theorem 1. *The class $\mathfrak{R}(m)$ of the m-dominant measurable sets of a conditionally outer measure μ defined in X, is μ-completely additive class.*

Proof. By Lemma 1, it is enough to show that $\mu(A) = m(A)$ for any set A which is an element of $\mathfrak{R}(m)$ and $A \subseteq K_n$ for some K_n. Then $m(K_n) = m(A) + m(K_n \frown CA)$, $\mu(K_n) \leqq \mu(A) + \mu(K_n \frown CA)$ and $m(K_n) = \mu(K_n)$ hold. Consequently $m(A) + m(K_n \frown CA) \leqq \mu(A) + \mu(K_n \frown CA)$. On the other hand, since $\mu(A) \leqq m(A)$ and $\mu(K_n \frown CA) \leqq \mu(K_n \frown CA)$, $m(A) + m(K_n \frown CA) \geqq \mu(A) + \mu(K_n \frown CA)$. Thus, $m(A) + m(K_n \frown CA) = \mu(A) + \mu(K_n \frown CA)$ and therefore $\mu(A) = m(A)$.

Corollary 2. *$\mu(A) = m(A)$ holds for any set A of the class $\mathfrak{R}(m)$ of m-dominant measurable sets of a conditionally finite outer measure μ defined in X.*

Theorem 2. *In order that a conditionally finite outer measure μ defined in X be relatively regular, it is necessary and sufficient that the class $\mathfrak{R}(\mu)$ of μ-modular sets be μ-completely additive class.*

Proof. i) when μ is relatively regular: There exists the regular outer measure μ_0 such that $\mu_0(A) \leq m_\alpha(A)$ for every dominant measure m_α of μ and μ is completely additive on $\mathfrak{R}(\mu_0)$ by Theorem 1. We shall show $\mathfrak{R}(\mu) = \mathfrak{R}(\mu_0)$. By Lemma 3, $\mathfrak{R}(\mu_0) \subseteq \mathfrak{R}(\mu)$ and therefore it is enough te see $\mathfrak{R}(\mu_0) \supseteq \mathfrak{R}(\mu)$. Suppose that there exists a set A such that $A \bar{\in} \mathfrak{R}(\mu_0)$ and $A \in \mathfrak{R}(\mu)$. Then, by the definition, we can suppose that A is a sub-set of some K_n. Since $\mu_0(K_n) < \mu_0(A) + \mu_0(K_n \frown CA)$, $\mu(K_n) = \mu(A) + \mu(K_n \frown CA)$ and $\mu_0(K_n) = \mu(K_n)$, then $\mu(A) + \mu(K_n \frown CA) < \mu_0(A) + \mu_0(K_n \frown CA)$ and at least one of $\mu(A) < \mu_0(A)$ and $\mu(K_n \frown CA) < \mu_0(K_n \frown CA)$ holds. On the other hand, there exists $\mathfrak{M}_\beta$ which satisfies $A \in \mathfrak{M}_\beta$ and $K_n \frown CA \in \mathfrak{M}_\beta$ by the Lemma 3. Let $m_\beta(A) = \inf \mu(B)$, where the infimum is taken over all the sets B such that $B \supseteq E$ and $B \in \mathfrak{M}_\beta$, then m_β is the dominant measure of μ by the extension theorem[4] and $\mathfrak{M}_\beta \geqq \mathfrak{C}(\mu)$. Moreover, at least one of $m_\beta(A) < \mu_0(A)$ and $m_\beta(K_n \frown CA) < \mu_0(K_n \frown CA)$ holds. This is contradict to the definition of μ_0. Therefore $\mathfrak{R}(\mu_0) \subseteq \mathfrak{R}(\mu)$. ii) When $\mathfrak{R}(\mu)$ is μ-completely additive class: If we put $m(A) = \inf \mu(B)$, where the infimum

4) E. Hopf: Ergodentheorie (1937), p. 2.

is taken over all sets B such that $B \supseteq A$ and $B \in \mathfrak{R}(\mu)$, by the extension theorem and $\mathfrak{R}(\mu) \supseteq \mathfrak{C}(\mu)$, m is the dominant measure of μ. Let $m_\beta(A)$ be an arbitrary dominant measure of μ. By Corollary 2, for any set A of $\mathfrak{R}(m_\beta)$ such that $A \subseteq K_n$ for some K_n, $\mu(A) = m_\beta(A)$ and $\mu(K_n \frown CA) = m_\beta(K_n \frown CA)$ and therefore, since $m_\beta(K_n) = m_\beta(A) + m_\beta(K_n \frown CA)$, $\mu(K_n) = \mu(A) + \mu(K_n \frown CA)$ holds. Thus, $A \in \mathfrak{R}(\mu)$ and also $\mathfrak{R}(m_\beta) \subseteq \mathfrak{R}(\mu)$. Using that m_β is regular and Corollary 2, $m_\beta(A) = \inf m_\beta(B) = \inf \mu(\beta)$, where the infimum is taken over all sets B such that $B \supseteq A$ and $B \in \mathfrak{R}(m_\beta)$, holds and moreover, since $\mathfrak{R}(m_\beta) \subseteq \mathfrak{R}(\mu)$ and the definition of m, we find that $m(A) \leqq m_\beta(A)$ and this completes the proof.

Corollary 3. *When a conditionally finite outer measure μ defined in X is regular, it is relatively regular*[5].

Remark 1. *There exist conditionally finite outer measures which are relatively regular, but non-regular measures.* **Example 1:** Let ν be the Lebesgue outer measure in the 2-dimensional Euclidean space and Ω a non-measurable set such that the inner measure is zero with $C\Omega$. When we put

$$\mu(A) = \nu(A) + \nu(A \frown \Omega), \tag{6}$$ [6]

$\mu(A)$ is non-regular outer measure. For the sub-set A of some K_n, we shall put

$$\mu_*(A) = \mu(K_n) - \mu(K_n \frown CA), \tag{7}$$

$$\nu_*(A) = \nu(K_n) - \nu(K_n \frown CA). \tag{8}$$

Now,
$$\begin{aligned} \mu_*(A) &= \mu(K_n) - \mu(K_n \frown CA) \\ &= \nu(K_n) + \nu(K_n \frown C\Omega) - \nu(K_n \frown CA) - \nu(K_n \frown CA \frown \Omega) \\ &= \nu_*(A) + \nu(K_n) - \nu(K_n \frown CA \frown \Omega) \\ &= \nu_*(A) + \nu_*(K_n \frown C(K_n \frown CA \frown \Omega)) = \nu_*(A) + \nu_*(A \smile C\Omega). \end{aligned}$$

Thus,

$$\mu_*(A) = \nu_*(A) + \nu_*(A \smile C\Omega) \tag{9}$$

Moreover,
$$\begin{aligned} \nu(A \frown \Omega) - \nu_*(A \smile C\Omega) &= \nu(A \frown \Omega) - \nu_*((A \frown \Omega) \smile C\Omega) \\ &\geqq \nu(A \frown \Omega) - \{\nu(A \frown \Omega) + \nu_*(C\Omega)\} = -\nu_*(C\Omega) = 0, \end{aligned}$$

whence

$$\nu(A \frown \Omega) \geqq \nu_*(A \smile C\Omega). \tag{10}$$

Using (6), (9) and (10), in order that $\mu(A) = \mu_*(A)$, it is necessary and sufficient that $\nu(A) = \nu_*(A)$ and $\nu(A \frown \Omega) = \nu_*(A \smile C\Omega)$.

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5) C. Carathéodory: Loc. cit. § 260.
6) C. Carathéodory: Loc. cit. § 339.

Let A be an element of $\mathfrak{N}(\nu)=\mathfrak{C}(\nu)$ such that $A\subseteq K_n$ for some K_n. Since $\nu(A)=\nu_*(A)$ and $\nu(A\frown\Omega)=\nu(A)$, $\nu_*(A\smile C\Omega)$ $=\nu_*(A\smile(C\Omega\frown CA))\geqq\nu_*(A)+\nu_*(C\Omega\frown CA)=\nu_*(A)=\nu(A)=\nu(A\frown\Omega)$ and therefore by (10), $\nu_*(A\smile C\Omega)=\nu(A\frown\Omega)$ follows. Consequently $\mathfrak{N}(\nu)=\mathfrak{C}(\nu)\subseteq\mathfrak{N}(\mu)$. On the other hand, since $\nu(A)=\nu_*(A)$ if $\mu(A)=\mu_*(A)$, $\mathfrak{N}(\mu)\subseteq\mathfrak{N}(\nu)$ and also

$$\mathfrak{N}(\mu)=\mathfrak{N}(\nu)=\mathfrak{C}(\nu)\,. \tag{11}$$

Let A be an element of $\mathfrak{C}(\nu)$ and W an arbitrary set. Then $\mu(W\frown A)+\mu(W\frown CA)=\nu(W\frown A)+\nu(W\frown A\frown\Omega)+\nu(W\frown CA)+\nu(W\frown$ $\frown CA\frown\Omega)=\nu(W)+\nu((W\frown\Omega)\frown A)+\nu((W\frown\Omega)\frown CA)=\nu(W)+\nu(W\frown\Omega)$ $=\mu(W)$. Therefore $\mathfrak{C}(\nu)\subseteq\mathfrak{C}(\mu)$ and namely, by (11),

$$\mathfrak{C}(\mu)=\mathfrak{N}(\mu)=\mathfrak{N}(\nu)=\mathfrak{C}(\nu)\,. \tag{12}$$

From the above consideration and Theorem 2, μ is relatively regular.

Remark 2. *There exist outer measures μ which are not relatively regular.* **Example 2:** Let ν and ν_* be the notation used in Example 1. Let us put[7]

$$\mu(A)=\frac{1}{2}(\nu(A)+\nu_*(A))\,. \tag{13}$$

Then $\mathfrak{N}(\mu)$ consists of every sub-sets of X. Because, for an arbitrary sub-set A of some K_n,

$$\begin{aligned}\mu(K_n)-\mu(A)&=\frac{1}{2}(\nu(K_n)+\nu_*(K_n))-\frac{1}{2}(\nu(A)+\nu_*(A))\\&=\frac{1}{2}(\nu(K_n)-\nu(A))+\frac{1}{2}(\nu(K_n)-\nu_*(A))\\&=\frac{1}{2}(\nu_*(K_n\frown CA)+\nu(K_n\frown CA))=\mu(K_n\frown CA).\end{aligned}$$

But there exist sets A_1, $A_2\in X$ such that

$$\mu(A_2)<\mu(A_2\frown A_1)+(A_2\frown CA_1)^{7)}\,.$$

Remark 3. *There exist outer measures μ for which there exists μ-completely additive class $\mathfrak{N}$ such that $\mathfrak{N}\supseteq\mathfrak{C}(\mu)$ and $\mathfrak{N}\neq\mathfrak{C}(\mu)$.* **Example 3:** Let μ be the outer measure given in Example 2. Then for the set A_1 given in Example 2, $A_1\in\mathfrak{N}(\mu)$, $A_1\bar{\in}\mathfrak{C}(\mu)$ and $[\mathfrak{C}(\mu), A_1]\neq\mathfrak{C}(\mu)$. Moreover, it is already shown in the proof of Lemma 3 that $[\mathfrak{C}(\mu), A_1]$ is μ-completely additive class.

7) C. Carathéodory: Loc. cit. § 605.