4. *A Lattice-Theoretic Treatment of Measures and Integrals.*

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In this paper we shall introduce a system of mathematical objects which is considered as a generalization of a set of Somen introduced by C. Carathéodory in his article [2], and which contains as particular cases a system of subsets of a set and a system of non-negative functions defined on a set. We shall give further the results corresponding to the theory of the Carathéodory's outer measure [1] and the extension theorem of Kolomogoroff-Hopf. [2]

1. In this section we shall deal with a mathematical object $\mathfrak{M}$ satisfying the following axioms.

Axiom 1. For every $A, B \in \mathfrak{M}$, one of the incompatible formulas $A=B$ or $A \neq B$ is accepted and "$=$" satisfies the following conditions:

(1.1) $A=A$; (1.2) If $A=B$, then $B=A$; (1.3) If $A=B$ and $B=C$, then $A=C$.

Axiom 2. For every $A, B \in \mathfrak{M}$, there exists only one element of $\mathfrak{M}$ denoted by $A \dotplus B$, satisfying the following conditions:

(2.1) $A \dotplus A=A$; (2.2) $A \dotplus B=B \dotplus A$; (2.3) $A \dotplus (B \dotplus C)=(A \dotplus B) \dotplus C$; (2.4) If $B=B'$, then $AB\dotplus =A \dotplus B'$.

$A \dotplus B$ will be called the *sum* of A and B.

Definition 1. If $A \dotplus B=A$, then B is said to be a *part* of A and denoted by $A \supseteq B$ or $B \subseteq A$.

Then $\mathfrak{M}$ may be regarded as an *ordered system* through the relation $A \supseteq B$ which we can replace by $A \geqq B$ and in this case Definition 1 should be taken as the definition of the enouncement "B is smaller than A" or "A is greater than B".

Axiom 3. For $\{A_n\}$, $A_n \in \mathfrak{M}$ [3], there exists the smallest element $V \in \mathfrak{M}$, of which every A_n is a part, and it will be written $V=A_1 \dotplus A_2 \dotplus \ldots$ or $V=\sum_{n=1}^{\infty} A_n$. V will be called the *sum* of $\{A_n\}$.

Axiom 4. There exists an element of $\mathfrak{M}$ which is a part of every element of $\mathfrak{M}$ and is called a *null element*.

Definition 2. For $A, B \in \mathfrak{M}$, if A and B has no common part except the null element, then we say that A and B are *disjunct* and write $A \circ B$ or $B \circ A$.

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1) Cf. [1].

2) Cf. [1].

3) A_n denotes A_n $(n=1, 2, \ldots)$.

Axiom 5. For B, $A_n \in \mathfrak{M}$, if B and A_n are disjunct for all n, then B and $\sum_{n=1}^{\infty} A_n$ are also disjunct.

Axiom 6. There exists a system $\mathfrak{N}$ of mathematical object related to $\mathfrak{M}$ satisfying the following conditions: (6.1) $\mathfrak{N}$ contains $\mathfrak{M}$ as a sub-system and satisfies Axioms 1, 2, 3, 4, and Axiom 5 either for $B \in \mathfrak{M}$, $A_n \in \mathfrak{N}$ or for $A_n \in \mathfrak{M}$, $B \in \mathfrak{N}$ and a null element of $\mathfrak{N}$ also belongs to $\mathfrak{M}$; (6.2) If A, $B \in \mathfrak{M} \subseteq \mathfrak{N}$, then the sum of them defined in $\mathfrak{N}$ has the same sense as in $\mathfrak{M}$; (6.3) For $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$, there exist uniquely $B_1, B_2 \in \mathfrak{N}$ such that $B_1 \dotplus B_2 = B$, $B_1 \circ A$ and $B_2 \subseteq A$ and that $B \in \mathfrak{M}$ implies $B_2 \in \mathfrak{M}$.

We call the elements B_1, B_2 *the components of B decomposed by A.*

There is exactly one common null element of $\mathfrak{M}$ and $\mathfrak{N}$, and each of them has no other null element.

Definition 3. If a system $\mathfrak{M}$ of mathematical objects satisfies Axioms 1, 2, 3, 4, 5, 6, then $\mathfrak{M}$ is said a *set of Somen* and an element of $\mathfrak{M}$ is said *Soma.*

In particular, if $\mathfrak{M} = \mathfrak{N}$, then $\mathfrak{M}$ coincides with a set of Somen defined by C. Carathéodory in [2]. In the sequel we shall denote sets of Somen by $\mathfrak{M}$, $\mathfrak{M}_1$, $\mathfrak{M}_m$ etc.

Lemma 1. 1) If A, $B \in \mathfrak{N}$, $A \subseteq B$ and $B \subseteq A$, then $A = B$. 2) For A, B, $C \in \mathfrak{N}$, $A \dotplus B \subseteq C$ if and only if $A \subseteq C$ and $B \subseteq C$. 3) If A, $B \in \mathfrak{N}$, $A \subseteq B$ and $A \circ B$, then $A = 0$. 4) If A, B, $C \in \mathfrak{N}$. $A \subseteq B$ and $B \circ C$, then $A \circ C$. 5) If $A_{kj} \in \mathfrak{N}$ ($k, j = 1, 2, \ldots$), then the sum of $\{A_{kj}\} = \sum_{k=1}^{\infty} (\sum_{j=1}^{\infty} A_{kj}) = \sum_{j=1}^{\infty} (\sum_{k=1}^{\infty} A_{kj})$.

Theorem 1. Let $A \in \mathfrak{M}$, $B \in \mathfrak{N}$ and the elements B_1, B_2 be the components of B decomposed by A such that $B_1 \circ A$ and $B_2 \subseteq A$, then it hold:

1) B_1 is the greatest part of B which is disjunct from A; 2) B_1 is the greatest part of B which is disjunct from B_2; 3) B_2 is the greatest common part of A and B; 4) B_2 is the greatest part of B which is disjunct from B_1.

Definition 4. For A, $B \in \mathfrak{N}$ if $A \circ B$, then we denote by $A + B$ the sum of A and B.

Definition 5. Let $A \in \mathfrak{M}$, $B \in \mathfrak{N}$ and the elements B_1, B_2 be the components of B decomposed by A such that $B_1 \circ A$ and $B_2 \subseteq A$, then we shall denote B_1 and B_2 by $B - BA$ and BA respectively.

Theorem 2. 1) If $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$, then $B = BA + (B - BA)$. 2) If A, $B \in \mathfrak{M}$, then $AB = BA$. 3) If A, $B \in \mathfrak{M}$ and $C \in \mathfrak{N}$, then $(CB)A = (CA)B = C(AB)$. If A, B, $C \in \mathfrak{M}$, then $A(BC) = (AB)C = B(AC)$ $(= ABC)$[4]. 4) If A, $B \in \mathfrak{M}$, then $A \dotplus B = AB + (A - AB) + (B - BA)$.

Theorem 3. Let it be $A_n \in \mathfrak{N}$, $B \in \mathfrak{M}$ and $V = \sum_{n=1}^{\infty} A_n$. Then we have $VB = \sum_{n=1}^{\infty} (A_n B)$ and $V - VB = \sum_{n=1}^{\infty} (A_n - A_n B)$.

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4) If we write ABC, we have no confusion.

Theorem 4. For $A, B, C \in \mathfrak{M}$, it holds:

$$(A-AB)C = (C-CB)A = AC-(AC)B.$$

Theorem 5. Let it be $E, F, A, A_n \in \mathfrak{M}$, then it hold:

1) $(E-EF)(\sum_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty}((E-EF)A_n)$;

2) $(E-EF)-(E-EF)A=E-E(A\dot{+}F)=(E-EA)-(E-EA)F$.

Definition 6. Let $\mathfrak{M}^*$ be the system which consists of the elements of $\mathfrak{M}$ and those of the form $A-AB$ for all $A, B \in \mathfrak{M}$.

We notice that we have UA, $U-UA \in \mathfrak{M}^*$ for any $U \in \mathfrak{M}^*$, $A \in \mathfrak{M}$.

Definition 7. Let $\mathfrak{M}$ and $\mathfrak{M}_1$ be two sets of Somen and α be a mapping of $\mathfrak{M}^*$ onto $\mathfrak{M}_1^*$, which is one to one onto $\mathfrak{M}_1$ when its domain is restricted to $\mathfrak{M}$. Suppose furthermore α satisfies the relations:

i) $\alpha(A)=\alpha(AB)\dot{+}\alpha(A-AB)$ for all $A, B \in \mathfrak{M}$;

ii) $\alpha(A-AB) \circ \alpha(B)$ for all $A, B \in \mathfrak{M}$.

Then, we say that α is an *isomorphic mapping of* $\mathfrak{M}$ *onto* $\mathfrak{M}_1$, and in the particular case $\mathfrak{M}_1=\mathfrak{M}$ we say that α is an *automorphic mapping of* $\mathfrak{M}$.

Theorem 6. Let α be an isomorphic mapping of $\mathfrak{M}$ onto $\mathfrak{M}_1$, then α is one to one onto $\mathfrak{M}_1^*$ and for $A, B, A_n \in \mathfrak{M}$ it hold:

1) $\alpha(AB)=\alpha(A)\alpha(B)$; 2) $\alpha(A-AB)=\alpha(A)-\alpha(A)\alpha(B)$; 3) $\alpha(A\dot{+}B)=\alpha(A)\dot{+}\alpha(B)$; 4) $\alpha(\sum_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty}\alpha(A_n)$; 5) $A\subseteq B$ holds if and only if $\alpha(A)\subseteq\alpha(B)$; 6) $\alpha((A-AB)A_1)=\alpha(A-AB)\alpha(A_1)$, $\alpha((A-AB)-(A-AB)A_1)=\alpha(A-AB)-\alpha(A-AB)\alpha(A_1)$.

2. *Definition 8.* Let μ be a functional defined on $\mathfrak{M}^*$ satisfying the following conditions:

i) $0\leq\mu(A)\leq+\infty$ for all $A\in\mathfrak{M}^*$; ii) If $A\subseteq B$ and $A, B\in\mathfrak{M}^*$, then $\mu(A)\leq\mu(B)$; iii) If $V\in\mathfrak{M}^*$, $A_n\in\mathfrak{M}$, $A_n\subseteq A_{n+1}$ and $V=\sum_{n=1}^{\infty}A_n$, then $\mu(UV)\leq\mu(UA_1)+\sum_{n=1}^{\infty}\mu(UA_{n+1}-UA_{n+1}A_n)+\mu(U-UV)$.

Then, we call μ an *outer measure defined on* $\mathfrak{M}$.

Definition 9. If $A\in\mathfrak{M}$ is an element such that

$$\mu(U)=\mu(UA)+\mu(U-UA) \qquad \text{for all } U\in\mathfrak{M}^*,$$

then A is said to be *μ-measurable.* By $\mathfrak{B}(\mu)$ we shall denote the family of all μ-measurable sets.

In order to show the measurablity of $A\in\mathfrak{M}$, it suffices to show that $\mu(E-EF)=\mu((E-EF)A)+\mu((E-EF)-(E-EF)A)$ for any $E, F\in\mathfrak{M}$ satisfying $\mu)E-EF)<+\infty$.

Theorem 7. $\mathfrak{B}(\mu)$ has the following properties:[5]

1) If $A, B\in\mathfrak{B}(\mu)$, then $A\dot{+}B\in\mathfrak{B}(\mu)$; 2) If $A, B\in\mathfrak{B}(\mu)$, then $AB\in\mathfrak{B}(\mu)$; 3) If $A, B\in\mathfrak{B}(\mu)$ and $A-AB\in\mathfrak{M}$, then $A-AB\in\mathfrak{B}(\mu)$;

5) Cf. [2].

4) If $A, B \in \mathfrak{B}(\mu)$ and $A \circ B$, then $\mu(A+B)=\mu(A)+\mu(B)$; 5) If $A_n \in \mathfrak{B}(\mu)$ and $A_n \subseteq A_{n+1}$, then $V=\sum_{n=1}^{\infty} A_n \in \mathfrak{B}(\mu)$ and $\lim_{n\to\infty} \mu(A_n)=\mu(V)$.

Theorem 8. Let α be an isomorphic mapping of $\mathfrak{M}$ onto $\mathfrak{M}_1$ and μ and μ_1 be outer measures in $\mathfrak{M}$ and $\mathfrak{M}_1$ respectively. If it hold:

$$\mu_1(\alpha(A)) = k(\mu(A)) \qquad (k \geqq 0) \qquad \text{for all } A \in \mathfrak{M}^*,$$

then μ-measurablity of A implies μ_1-measurablity of $\alpha(A)$.

3. *Definition 10.* Let $\mathfrak{B}$ be a sub-system of $\mathfrak{M}$ satisfying the following conditions:

i) If A and B belong to $\mathfrak{R}$, then so do $A \dotplus B$ and AB; ii) If A_n belongs to $\mathfrak{B}$ and $A_n \subseteq A_{n+1}$ for all n, then $\sum_{n=1}^{\infty} A_n \in \mathfrak{B}$. Then, $\mathfrak{B}$ is called *Baire set*. We denote by $\mathfrak{B}(\mathfrak{S})$ the smallest Baire set containing a sub-system $\mathfrak{S}$ of $\mathfrak{M}$.

Theorem 9. Let $\mathfrak{S}$ be a sub-system of $\mathfrak{M}$ satisfying the following conditions:

1) If A and B belong to $\mathfrak{S}$, then so do $A \dotplus B$ and AB; 2) A functional m is defined on $\mathfrak{S}^*$, which consists of $\mathfrak{S}$ and those of the form $A-AB$ for all $A, B \in \mathfrak{S}$, and satisfies that $0 \leqq m(A) < +\infty$ for all $A \in \mathfrak{S}^*$ and that $m(U)=m(UB)+m(U-UB)$ for all $U \in \mathfrak{S}^*$ and $B \in \mathfrak{S}$; 3) Let it be $U_n \in \mathfrak{S}^*$, $U_n \subseteq U_{n+1}$ and $A=\sum_{n=1}^{\infty} U_n \in \mathfrak{S}$, then $m(A) \leqq \lim_{n\to\infty} m(U_n)$.

Then, there exists a functional $\mu(A)$ defined on $\mathfrak{B}(\mathfrak{S})$ having the following properties:

1′) $0 \leqq \mu(A) < +\infty$ for all $A \in \mathfrak{B}(\mathfrak{S})$; 2′) If $A, B \in \mathfrak{B}(\mathfrak{S})$ are disjunct, then $\mu(A+B)=\mu(A)+\mu(B)$; 3′) If $A \in \mathfrak{S}$, then $\mu(A)=m(A)$; 4′) If $A_n \in \mathfrak{B}(\mu)$ and $A_n \subseteq A_{n+1}$, then $\lim_{n\to\infty} \mu(A_n)=\mu(\sum_{n=1}^{\infty} A_n)$.

Corollary 1. Let it be $\mathfrak{S}$ and m as in Theorem 9. Besides, if $\mathfrak{M}^*=\mathfrak{M}$ then there exists a functional μ defined on the smallest Baire set $\mathfrak{B}_c(\mathfrak{S})$ such that $A-AB \in \mathfrak{B}_c(\mathfrak{S})$ for all $A, B \in \mathfrak{B}_c(\mathfrak{S})$ and that $\mathfrak{B}_c(\mathfrak{S}) \supseteq \mathfrak{S}$, and satisfying 1′), 2′), 3′), 4′) in Theorem 9 for $\mathfrak{B}_c(\mathfrak{S})$ in stead of $\mathfrak{B}(\mathfrak{S})$.[6]

Theorem 10. Let it be $\mathfrak{S}$ and m as in Theorem 9 and α be an automorphic mapping of $\mathfrak{M}$. If α is a mapping of $\mathfrak{S}$ onto itself and if $m(\alpha(A))=k(m(A))$ $(k \geqq 0)$ for all $A \in \mathfrak{S}$. Then, α is also a mapping of $\mathfrak{B}(\mathfrak{S})$ onto itself and $\mu(\alpha(A))=k(m(A))$ for all $A \in \mathfrak{B}(\mathfrak{S})$.

Applications of the above theory to a system of subsets of a set and to a system of non-negative functions defined on a set will be given in another article.[7]

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6) Cf. [3].

7) Cf. [4], [5], [6], [7], [8].

References.

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