46. *On a Fundamental Lemma on Weakly Normal Rings*

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Let R be an (associative) ring. With a subset X of R we denote by X_r (resp. X_l) the set of right (resp. left) multiplications of the elements of X onto R. The commuter $V_{\mathfrak{A}}(X_r)$ of X_r in the absolute (module-) endomorphism ring $\mathfrak{A}$ of R is nothing but the X-right-endomorphism ring of R. Now, if S is a subring of R and if the S_r-endomorphism ring $V_{\mathfrak{A}}(S_r)$ of R (which certainly contains R_l) is generated over R_l by a family of R_l-semilinear endomorphisms of R, then we say that S is a weakly normal[*)] subring of R. Recently the writer studied the case where the ring R and its weakly normal subring S are simple rings with minimum condition (or complete primitive rings[**)]) and showed that then R is fully reducible as an R_l S_r-module[8)]; this enabled the writer to obtain a theorem of extension of isomorphisms of certain weakly normal subrings, which forms a generalization and a refinement of the theorems of Artin-Whaples[1)] and Cartan-Dieudonné[4)], to establish a simple ring generalization of the Cartan-Jacobson[3)6)] Galois theory (for sfields), and further, to extend Hochschild's[5)] cohomology theory of simple algebras to simple rings[8)9)10)]. The purpose of the present short note is to observe that this fundamental lemma remains true also in case the subring S is not necessarily simple (or complete primitive) but merely semisimple. This extension entails a corresponding generalization in cohomology theory and has some bearings for Galois theory, though we shall not discuss these in the present note.

We prove thus

Theorem 1 (*Fundamental lemma*). *Let R be a simple ring having unit element* 1 *and satisfying minimum condition. Let S be a weakly normal semisimple subring of R containing* 1 *and satisfying minimum condition. Then R is fully reducible as an R-left- and S-right-module.*

Proof. Evidently R is S_r-fully reducible. Let

$$R = \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \cdots \oplus \mathfrak{N}_s$$

be the idealistic decomposition of the S_r-module R; thus each $\mathfrak{N}_i$ is homogeneously fully reducible with respect to S_r, and distinct $\mathfrak{N}_i$, $\mathfrak{N}_j$ have no mutually isomorphic minimal S_r-submodules. The S_r-endo-

*) Dieudonné 4), Nakayama 8)9)10).

**) With certain modification of definition and under certain restrictions.

morphism ring $V_{\mathfrak{A}}(S_r)$ of R is then the direct sum

$$V_{\mathfrak{A}}(S_r) = V_1 \oplus V_2 \oplus \cdots \oplus V_s,$$

where each V_i is isomorphic, and may be identified, with the S_r-endomorphism ring of $\mathfrak{N}_i$. Each $\mathfrak{N}_i$ is V_i S_r-minimal*), or what is the same, $V_{\mathfrak{A}}(S_r)S_r$-minimal.

Let $\mathfrak{n}$ be a minimal S_r-submodule of $\mathfrak{N}_1$, say. For any $a_l \in R_l$, $\mathfrak{n}^{a_l}$ is S_r-submodule of R which is either 0 or S_r-isomorphic with $\mathfrak{n}$. So $\mathfrak{n}^{a_l} \subseteq \mathfrak{N}_1$. Hence $\mathfrak{N}_1$, and similarly each $\mathfrak{N}_i$, is R_lS_r-allowable. Each $\mathfrak{N}_i$ has, together with R, an R_l-composition series, whence an R_lS_r-composition series. Take, for each i, a minimal R_lS_r-submodule $\mathfrak{m}_i$ of $\mathfrak{N}_i$

As S is, by our assumption, weakly normal in R, we have

$$V_{\mathfrak{A}}(S_r) = \sum R_l \gamma$$

with some R_l-semilinear endomorphisms γ of R. Consider the sum $\sum_{\gamma} \mathfrak{m}_1^{\gamma}$; each $\mathfrak{m}_1^{\gamma}$ is R_l-semilinearly and S_r-linearly isomorphic to $\mathfrak{m}_1$ and is thus R_lS_r-minimal. It is $V_{\mathfrak{A}}(S_r)S_r$-allowable. Clearly $\sum_{\gamma}\mathfrak{m}_1^{\gamma} \frown \mathfrak{N} \neq 0$. As, on the other hand, $\mathfrak{N}_1$ is $V_{\mathfrak{A}}(S_r)S_r$-minimal, we have $\sum_{\gamma}\mathfrak{m}_1^{\gamma} \supseteq \mathfrak{N}_1$. Hence $\mathfrak{N}_1$, together with $\sum_{\gamma}\mathfrak{m}_1^{\gamma}$, is R_lS_r-fully reducible (and is a direct sum of mutually R_l-semilinearly and S_r-linearly isomorphic minimal R_lS_r-submodules). Similarly each $\mathfrak{N}_i$ is R_lS_r-fully reducible, and our theorem is proved.

It follows in particular that the commuter $V_R(S)$ of S in R (with R, S as in Theorem 1), which is isomorphic to the R_lS_r-endomorphism ring of R, is a semisimple ring with minimum condition.

When we deal, as above, with a simple ring R with unit element and with minimum condition, the above definition of the weak normality of a subring is equivalent to that the S-endomorphism ring of a minimal right-ideal $\mathfrak{r}$ of R is generated over the R-endomorphism sfield K of $\mathfrak{r}$ by a family of K-semilinear endomorphisms. For an ideal-primitive ring**) R we employ this last definition of weak normality***) (on making use of a minimal right-ideal $\mathfrak{r}$ of R which is determined uniquely up to isomorphisms). Then we have

Theorem 1′. *Let R be an ideal-primitive ring. Let $\mathfrak{r}$ be a minimal right-ideal of R and K be its R-endomorphism sfield. Let S be a weakly normal distinguished****) semi-primitive*****) subring of R. Suppose that every none-zero KS-submodule of $\mathfrak{r}$ contains a minimal KS-submodule. Then $\mathfrak{r}$ is fully reducible as a KS-module (or, equivalently, the (unique) smallest two-sided ideal $\mathfrak{z}$ of R is fully reducible as an R-left- and S-right-module).*

Proof runs similarly as above, if $\mathfrak{r}$, K are considered in place of R, R_l.

*) If a module $\mathfrak{m}$ is homogeneously fully reducible with respect to an operator domain $\mathfrak{S}$ and if $\mathfrak{T}$ is the $\mathfrak{S}$-endomorphism ring of $\mathfrak{m}$, then $\mathfrak{m}$ is $\mathfrak{S}\mathfrak{T}$-minimal; see 8), § 1.

**) A primitive ring we call ideal-primitive, when it possesses a faithful minimal right-ideal; then it possesses a faithful minimal left-ideal too 2)7).

***) We could use in place of $\mathfrak{r}$ the (unique) smallest two-sided ideal $\mathfrak{z}$ of R (and in place of K the left-multiplication of R on $\mathfrak{z}$).

****) We mean by this that $\mathfrak{r}$ is a fully reducible right-module of the subring.

*****) I. e. "semisimple" in Jacobson's sense.

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