70. On Mixed Boundary Value Problems for a Circle

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An explicit formula for the solution of a mixed boundary value problem in potential theory has recently been given by one of the present authors in the simplest case where the basic domain is the unit circle and there are two arcs on the circumference along which the values of the required function itself and of its normal derivative are prescribed[1]. The problem is formulated as follows: To determine a function $u(z)$ harmonic and bounded in the unit circle $|z|<1$ and satisfying the boundary conditions

$$u(e^{i\varphi})=U(\varphi) \text{ for } a<\varphi<b\,, \quad \frac{\partial u(e^{i\varphi})}{\partial \nu}=V(\varphi) \text{ for } b<\varphi<a+2\pi\,,$$

$\partial/\partial\nu$ denoting the differentiation along the inward normal at $e^{i\varphi}$.

The previous expression has been derived, as an illustration of the general discussion developed there, with the aid of a slit mapping function. In the present Note we shall first show that an equivalent formula can be derived in another way. Our present method of attack is based on a decomposition of the solution into two harmonic functions, of which the one solves a Dirichlet problem and the other a Neumann problem, and it applies efficiently also for the case where there are several arcs on the circumference of the unit circle along which the values of the function itself and of its normal derivative are alternately prescribed. The case of two pairs of boundary arcs will be explicitly treated in the latter part of the present Note.

Now, the solution $u(z)$ of the simplest problem formulated above may be regarded as the superposition of $u^{(1)}(z)$ and $u^{(2)}(z)$, i.e. $u(z)=u^{(1)}(z)+u^{(2)}(z)$, where $u^{(1)}(z)$ and $u^{(2)}(z)$ solve the problems with the special boundary conditions

$$\begin{aligned} &u^{(1)}(e^{i\varphi}) = U(\varphi) \text{ and } && u^{(2)}(e^{i\varphi}) = 0 && \text{for } a<\varphi<b\,,\\ &\frac{\partial u^{(1)}(e^{i\varphi})}{\partial\nu}=0 \text{ and } && \frac{\partial u^{(2)}(e^{i\varphi})}{\partial\nu}=V(\varphi) && \text{for } b<\varphi<a+2\pi\,.\end{aligned}$$

Based on the special boundary character, the problems of deter-

1) Y. Komatu, Mixed boundary value problems. Journ. Fac. Sci. Univ. Tokyo **6** (1953), 345-391; cf. also a preparatory announcement made in Y. Komatu, Eine gemischte Randwertaufgabe für einen Kreis. Proc. Japan Acad. **28** (1952), 339-341. The general problem of this type has once discussed also by A. Signorini, Sopra un problema al contorno nella teoria delle funzioni di variabile complessa. Ann. Mat. Pura Appl. (3) **25** (1916), 253-273.

mining $u^{(1)}(z)$ and $u^{(2)}(z)$ are reducible to those of solving respectively a Dirichlet as well as a Neumann problem for the unit circle. In fact, we map the unit circle $|z|<1$ onto the upper semicircle $|w|<1$, $\mathfrak{J}w>0$ in such a manner that the points $z=e^{ia}$ and $z=e^{ib}$ correspond to $w=+1$ and $w=-1$, respectively. Such a mapping function $z=z^{(1)}(w)$ is defined by

$$\frac{w+1}{w-1}=-e^{-i(b-a)/4}\frac{\sqrt{z-e^{ib}}}{\sqrt{z-e^{ia}}},$$

the square root $\sqrt{z-e^{ic}}$ ($c=a$ or $c=b$) denoting the branch which attains the value $ie^{ic/2}$ at $z=0$. The function $u^{(1)}(z)$ is then transformed into $u^{(1)*}(w)\equiv u^{(1)}(z^{(1)}(w))$ harmonic in the upper semicircle and satisfying the boundary conditions

$$u^{(1)*}(e^{i\varphi})=U(\varphi) \text{ for } 0<\psi<\pi,\ \frac{\partial u^{(1)*}(w)}{\partial\nu}=0 \text{ for } \mathfrak{J}w=0,\ |w|<1,$$

where $e^{i\psi}$ denotes the image of $e^{i\varphi}$ with $a<\varphi<b$: $e^{i\varphi}=z^{(1)}(e^{i\psi})$. The function $u^{(1)*}(w)$ being prolongable harmonically into the lower semicircle by means of the defining equation $u^{(1)*}(w)=u^{(1)*}(\overline{w})$, we can apply the Poisson integral representation, obtaining the expression

$$u^{(1)*}(w)=\Re\frac{1}{2\pi}\int_0^{\pi}u^{(1)*}(e^{i\psi})\left(\frac{e^{i\psi}+w}{e^{i\psi}-w}+\frac{e^{-i\psi}+w}{e^{-i\psi}-w}\right)d\psi.$$

We next map the unit circle $|z|<1$ onto the lower semicircle $|w|<1$, $\mathfrak{J}w<0$ in such a manner that the points $z=e^{ib}$ and $z=e^{ia}$ $(=e^{i(a+2\pi)})$ correspond to $w=-1$ and $w=+1$, respectively. Such a mapping function $z=z^{(2)}(w)$ is defined by

$$\frac{w+1}{w-1}=ie^{-i(b-a)/4}\frac{\sqrt{z-e^{ib}}}{\sqrt{z-e^{ia}}},$$

the square root denoting the same branch as above The function $u^{(2)}(z)$ is then transformed into $u^{(2)*}(w)\equiv u^{(2)}(z^{(2)}(w))$ harmonic in the lower semicircle and satisfying the boundary conditions

$$u^{(2)*}(w)=0 \text{ for } \mathfrak{J}w=0,\ |w|<1,\ \frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu}d\psi=V(\varphi)d\varphi \text{ for } \pi<\psi<2\pi,$$

where $e^{i\psi}$ denotes here the image of $e^{i\varphi}$ with $b<\varphi<a+2\pi$: $e^{i\varphi}=z^{(2)}(e^{i\psi})$. The function $u^{(2)*}(w)$ being prolongable into the upper semicircle by means of the defining equation $u^{(2)*}(w)=-u^{(2)*}(\overline{w})$, we can apply the well-known integral representation for the Neumann problem, obtaining the expression

$$u^{(2)*}(w)=-\Re\frac{1}{\pi}\int_{\pi}^{2\pi}\lg\frac{1-e^{i\psi}w}{e^{i\psi}-w}\cdot\frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu}d\psi;$$

an additive constant vanishes here in view of $u^{(2)*}(0)=0$.

Thus, both component-functions having been explicitly expressed, the formula for the solution of the original problem is derived by merely returning to the original variable. Actual computation will lead to the following result.

Theorem 1. *The mixed boundary value problem*

$$\Delta u(z)=0 \quad in\ |z|<1,$$

$$u(e^{i\varphi})=U(\varphi)\ for\ a<\varphi<b,\ \frac{\partial u(e^{i\varphi})}{\partial\nu}=V(\varphi)\ for\ b<\varphi<a+2\pi$$

is solved by the formula

$$u(z)=\Re\Bigg\{\frac{1}{2\pi}\int_a^b U(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi$$

$$-\frac{1}{\pi}\int_b^{a+2\pi}V(\varphi)$$

$$\times\lg\frac{\left(e^{i(b-a)/8}\left(\sin\frac{\varphi-b}{2}\right)^{1/2}\sqrt{z-e^{ia}}+e^{-i(b-a)/8}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}\sqrt{z-e^{ib}}\right)^2}{\sin\frac{b-a}{2}\cdot(z-e^{i\varphi})}d\varphi\Bigg\},$$

the square root $\sqrt{z-e^{ic}}$ denoting the branch which attains the value $ie^{ic/2}$ at $z=0$. More precisely stated, if $U(\varphi)$ and $V(\varphi)$ are continuous and bounded in their respective intervals of definition, the function $u(z)$ defined by the formula is harmonic and bounded in $|z|<1$ and satisfies the assigned boundary conditions. It will further be verified that, if $U(\varphi)$ and $V(\varphi)$ are any functions integrable, together with $U(\varphi)/\sqrt{(\varphi-a)(b-\varphi)}$, over their respective intervals, then the functions defined by the formula is harmonic in $|z|<1$ and satisfies the boundary conditions almost everywhere.

The formula now derived is apparently different in form with the one obtained in the previous paper. But, the identification will be confirmed by transforming the one into another by actual calculation.

We next proceed to deal with the problem where there are two pairs of the boundary arcs bearing the prescribed boundary values of the function itself and of its normal derivative. In general, under any conformal mapping, the boundary values of the function itself as well as of its normal derivative multiplied by the arc-element are conformally invariant. On the other hand, the unit circle can be mapped onto a rectangle in such a manner that any four preassigned points on the circumference correspond to the vertices of the rectangle.

In view of the circumstance just explained, we may choose, instead of the unit circle, a rectangle as the basic domain. Let it be defined by

$$\lg q<\Re z<0, \qquad 0<\Im z<\pi,$$

q being a positive parameter less than unity. The problem is then formulated as follows: To determine a function $u(z)$ harmonic and bounded in the rectangle and satisfying the boundary conditions

$$u(it) = M(t) \text{ and } u(\lg q + it) = N(t) \text{ for } 0 < t < \pi,$$
$$\frac{\partial u(s)}{\partial \nu} = P(s) \text{ and } \frac{\partial u(s+i\pi)}{\partial \nu} = Q(s) \text{ for } \lg q < s < 0,$$

$\partial/\partial\nu$ denoting the differentiation along the inward normal at the respective points.

Quite similarly as in the simplest case, the solution of the present problem may be regarded as the superposition of two bounded harmonic functions $u^{(1)}(z)$ and $u^{(2)}(z)$, where $u^{(1)}(z)$ and $u^{(2)}(z)$ satisfy the same boundary conditions as $u(z)$ along vertical and horizontal boundary sides, respectively, while $\partial u^{(1)}(z)/\partial\nu$ and $u^{(2)}(z)$ vanish along horizontal and vertical boundary sides, respectively. The problems of determining $u^{(1)}(z)$ and $u^{(2)}(z)$ are then reducible to those of solving respectively a Dirichlet as well as Neumann problem for the annuli. In fact, the basic rectangle is mapped by

$$w = e^z$$

onto the upper semi-annulus $q<|w|<1$, $\Im w>0$ in such a manner that the vertices correspond to the angular points of the semi-annulus. The function $u^{(1)}(z)$ is then transformed into $u^{(1)}(\lg w)$ harmonic in the upper semi-annulus and satisfying the boundary conditions

$$u^{(1)}(\lg e^{i\psi}) = M(\psi) \text{ and } u^{(1)}(\lg(qe^{i\psi})) = N(\psi) \text{ for } 0<\psi<\pi,$$
$$\frac{\partial u^{(1)}(\lg(\pm e^s))}{\partial \nu} = 0 \quad \text{for } \lg q < s < 0,$$

$\partial/\partial\nu$ denoting here the differentiation along the inward normal of the semi-annulus at $w=\pm e^s$. The function $u^{(1)}(\lg w)$ being prolongable into the lower semi-annulus by means of the defining equation $u^{(1)}(\lg w)=u^{(1)}(\lg \overline{w})$, we can apply the Villat integral representation, obtaining the expression

$$\begin{aligned} u^{(1)}(\lg w) = \Re\Big\{ \frac{1}{\pi i}\Big(\frac{2\eta_3 \lg w}{\lg q}\int_0^\pi (M(\psi)-N(\psi))d\psi \\ + \int_0^\pi (M(\psi)(\zeta(i\lg w+\psi)+\zeta(i\lg w-\psi)) \\ - N(\psi)(\zeta_3(i\lg w+\psi)+\zeta_3(i\lg w-\psi)))d\psi\Big)\Big\}, \end{aligned}$$

the notations from the Weierstrassian theory of elliptic functions referring to those with the primitive periods $2\omega_1=2\pi$, $2\omega_3=-2i\lg q$.

Next, the basic rectangle is mapped by

$$w = e^{-i\pi z/\lg q}$$

onto the lower semi-annulus $e^{\pi^2/\lg q}<|w|<1$, $\Im w<0$ in such a manner that the vertices correspond to the angular points of the semi-annulus. The function $u^{(2)}(z)$ is then transformed into $u^{(2)}(i\lg q\cdot\lg w/\pi)$ harmonic in the lower semi-annulus and satisfying the boundary conditions

$$u^{(2)}(i\lg q\cdot\lg(\pm e^{\pi t/\lg q})/\pi)=0 \quad \text{for } 0<t<\pi,$$

$$\frac{\partial u^{(2)}(e^{i\psi})}{\partial\nu}=-\frac{\lg q}{\pi}P\Big(-\frac{\lg q}{\pi}\psi\Big) \text{ and}$$

$$\frac{\partial u^{(2)}(e^{\pi^2/\lg q+i\psi})}{\partial\nu}=-\frac{\lg q}{\pi}e^{-\pi^2/\lg q}Q\Big(-\frac{\lg q}{\pi}\psi\Big) \text{ for } -\pi<\psi<0.$$

The function $u^{(2)}(i\lg q\cdot\lg w/\pi)$ being prolongable into the upper semi-annulus by means of the defining equation $u^{(2)}(i\lg q\cdot\lg w/\pi) = -u^{(2)}(i\lg q\cdot\lg \overline{w}/\pi)$, we can apply an integral representation for the Neumann problem[2], obtaining the expression

$$u^{(2)}(i g q\cdot\lg w/\pi)=\Re\Big\{\frac{\lg q}{\pi^2}\Big(\frac{2\hat{\eta}_1 i\lg w}{\pi}\int_{-\pi}^{0}\psi\Big(P\Big(-\frac{\lg q}{\pi}\psi\Big)+Q\Big(-\frac{\lg q}{\pi}\psi\Big)\Big)d\psi$$

$$-\int_{-\pi}^{0}\Big(\lg\frac{\hat{\sigma}(i\lg w+\psi)}{\hat{\sigma}(i\lg w-\psi)}\cdot P\Big(-\frac{\lg q}{\pi}\psi\Big)+\lg\frac{\hat{\sigma}_3(i\lg w+\psi)}{\hat{\sigma}_3(i\lg w-\psi)}\cdot Q\Big(-\frac{\lg q}{\pi}\psi\Big)\Big)d\psi\Big)\Big\},$$

the notations marked by ^ depending on the primitive periods $2\hat{\omega}_1 =2\pi$, $2\hat{\omega}_3=-2i\pi^2/\lg q$; an additive constant vanishes in view of the antisymmetry character of the boundary functions.

Thus, both component-functions having been explicitly expressed, the formula for the solution of the original problem is derived by returning to the original variable. Actual computation will lead to the following result.

Theorem 2. *The mixed boundary value problem*

$$\Delta u(z)=0 \quad \textit{in} \quad \lg q<\Re z<0,\ 0<\Im z<\pi,$$

$$u(it)=M(t) \textit{ and } u(\lg q+it)=N(t) \textit{ for } 0<t<\pi,$$

$$\frac{\partial u(s)}{\partial\nu}=P(s) \textit{ and } \frac{\partial u(s+i\pi)}{\partial\nu}=Q(s) \textit{ for } \lg q<s<0$$

is solved by the formula

$$u(z)=\Re\Big\{\frac{2\eta_3 z}{\pi i\lg q}\Big(\int_0^\pi(M(t)-N(t))dt+\int_{\lg q}^0 s(P(s)+Q(s))\,ds\Big)$$

$$+\frac{1}{\pi i}\int_0^\pi(M(t)(\zeta(iz+t)+\zeta(iz-t))-N(t)(\zeta_3(iz+t)+\zeta_3(iz-t)))\,dt$$

$$+\frac{1}{\pi}\int_{\lg q}^0\Big(P(s)\lg\frac{\sigma(iz-is)}{\sigma(iz+is)}+Q(s)\lg\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}\Big)ds\Big\},$$

the notations from the Weierstrassian theory of elliptic functions referring to those with the primitive periods $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$. More precisely stated, if $M(t)$, $N(t)$, $P(s)$, and $Q(s)$ are continuous and bounded in their respective intervals of definition, the function $u(z)$ defined by the formula is harmonic and bounded in the basic rectangle and satisfies the assigned boundary conditions. It will further

2) Cf. Y. Komatu, Integralformel betreffend Neumannsche Randwertaufgabe für einen Kreisring. Kōdai Math. Sem. Rep. (1953), 37-40.

be verified that, if $M(t)$, $N(t)$, $P(s)$, and $Q(s)$ are any functions integrable over their respective intervals, then the function $u(z)$ defined by the formula is harmonic in the rectangle and satisfies the boundary conditions almost everywhere.

The details of the present preparatory announcement, together with an extension to general case, will be soon published in Kōdai Mathematical Seminar Reports.