223. On Imbeddings and Colorings of Graphs. I

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§1. **Introduction.** A graph G is an ordered pair (G^0, G^1), where G^0 is a nonempty finite set of objects and G^1 is a set of unordered finite pairs of the elements of G^0, where G^1 can contain some pairs of the same elements of G^0. The objects in G^0 and G^1 are called *vertices* and *arcs* of the graph G, respectively.

For two graphs $G=(G^0, G^1)$ and $H=(H^0, H^1)$, H is called a subgraph of G and noted by $G\supset H$, if $G^0\supset H^0$ and $G^1\supset H^1$.

We can realize any graph by an 1-dimensional complex, where we assume that an arc is an open 1-simplex in the complex. A graph is *connected* when it is connected as a complex. In this paper, a graph implies a connected graph.

A subset δ of G^0 is called *SC-set* (same-colorable set), if for any two vertices A and B in δ, the pair (A, B) is not contained in G^1.

A graph $G=(G^0, G^1)$ is *n-colorable* if such n SC-sets $\delta_1, \cdots, \delta_n$ exist as $G^0=\delta_1\cup\cdots\cup\delta_n$, $\delta_i\neq\phi$ and if $i\neq j$ $\delta_i\neq\delta_j$ for $i, j=1, \cdots, n$.

G is *n-chromatic* (or the *chromatic number* of G is n), if G is n-colorable but not n'-colorable for any $n'<n$.

The definition of n-colorable graphs in this paper is distinct from the one in [3],*) but the definitions of n-chromatic graphs in this paper and in [3] are equivalent.

In this paper a surface means a differentiable or combinatorial closed 2-manifold and an imbedding of G into a surface M means differentiable or piece-wise linear one, regarding G as a complex. For definition of differentiable map of complex, see, for example, [2].

G can be imbedded in some orientable surface having enough large number of genus. G is *m-imbeddable* if it can be imbedded in an orientable surface having genus m, and is *minimal m-imbeddable* if it can be m-imbeddable but not $(m-1)$-imbeddable. It is said that the *genus* of G is m if G is minimal m-imbeddable.

§2. **An imbedding theorem.** To express an imbedding of G into a surface M or the imbedded subspace of M, we use the notation $G(M)$. An imbedding $G(M)$ is said to be *simplest* if $\chi(M)\geqq\chi(N)$ for any imbedding $G(N)$, where $\chi(M)$ is the Euler characteristic of M. If any connected component of $M-G(M)$ is open 2-cell, $G(M)$ is said to be *2-cell imbedding*.

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*) See the references on the last page (p. 1024) of the part II of this paper.

For a component α of $M-G(M)$ and an arc a of $G(M)\cap Bd\ \alpha$, where $Bd\ \alpha$ is a boundary of α, a function $l(a, \alpha)$ is given by,

$$l(a, \alpha)=1, \quad \text{if } \mathrm{In}\,(\mathrm{Cl}\,\alpha)\not\supset a$$
$$=2, \quad \text{if } \mathrm{In}\,(\mathrm{Cl}\,\alpha)\supset a.$$

Then, α is said to be *l-gon* when $l=\sum_{a\in \mathrm{Cl}\alpha} l(a, \alpha)$.

$G(M)$ is said to be *l-gon imbedding*, if (i) $G(M)$ is a 2-cell imbedding, (ii) any component of $M-G(M)$ is a l-gon, and (iii) the number of the components of $M-G(M)$ is larger than or equal to 2.

The next theorem can be proved by extensive use of J. W. T. Youngs' method ([4]), in the case of $l=3$, who proves the theorem.

Theorem (2.1). *Let G have no k-circuit with $k<l$, and $G(M)$ be an l-gon imbedding, then $G(M)$ is a simplest imbedding; moreover if $G(N)$ is an imbedding in a surface N with the same Euler characteristic as M, then $G(N)$ is the l-gon imbedding.*

§ 3. Main theorem.

Lemma (3.1). *If $H\subset G$, the chromatic number of H is not larger than the one of G.*

Proof. Let n be the chromatic number of G. As G is n-colorble, $G^0=\delta_1\cup\cdots\cup\delta_n$, where δ_i is an SC-set. If we put $\delta'_i=\delta_i\cap H^0(i=1, \cdots, n)$, $H^0=\delta'_i\cup\cdots\cup\delta'_n$, and δ'_i is an SC-set on H. As it can be happen that $\delta'_i=\delta'_j$ for $i=j$ $H^0=\delta''_1\cup\cdots\cup\delta''_m$ $(m\leqq n)$ where δ''_1 is an SC-set.

Let $G=(G^0, G^1)$, then $G^0=\gamma_1\cup\cdots\cup\gamma_n$ is said to be a *color-classification* and γ_i a *color-class* of G, if any γ_i is an SC-set, $\gamma_i\neq\phi$ for any i and $\gamma_i\cap\gamma_j=\phi$ for $i\neq j$.

Generally, let $G(M)$ be a 2-cell imbedding and α be a connected component of $M-G(M)$. Let a closed disk D^2 be the subspace of euclidean space of dimension 2 consisting of points (x_1, x_2) such that $x_1^2+x_2^2\leqq 1$. Let $\varphi_\alpha: D^2\to \mathrm{Cl}\,\alpha$ be a continuous map from a closed 2-disk into closure of α such that (i) $\varphi_\alpha\,|\,\mathrm{In}\,D^2$ is a homeomorphism $\mathrm{In}\,D^2\to\alpha$, (ii) $\varphi_\alpha(Bd\,D^2)=Bd\,\alpha$ and (iii) when α is l-gon, there is points $P_1, \cdots, P_l$ in $Bd\ D$ such that $\varphi_\alpha\,|\,(P_i, P_{i+1})$ is a homeomorphism of (P_i, P_{i+1}) onto (A, B), where (P_i, P_{i+1}) is an arc between the adjoining points on $S^1=Bd\,D^2$ and (A, B) is an arc of G on $Bd\,\alpha$. Then, these points $P_1, \cdots, P_l$ are called the *vertices of D^2* related to φ_α and φ_α is called a *polygonal representation* of α. For any α, there is a polygonal representation φ_α. And, moreover, the polygonal representation of α has the properties that for any two polygonal representations $\varphi_\alpha, \varphi'_\alpha$ of α, there are orientations on $S^1=Bd\,D^2$ and vertices of $D^2, P_1\cdots, P_l;\ P'_1, \cdots, P'_l$ related respectively to φ_α and φ'_α which are ordered in the direction of each orientation and $\varphi_\alpha(P_i)=\varphi'_\alpha(p'), i=1, \cdots, l$.

Lemma (3.2). *If H has no k-circuit with $k<3$, the chromatic number of H is $\geqq 3$ and if there exists 2-cell imbedding $H(M)$, then there is a graph G such that* (i) *the chromatic number of G is equal to that of H,* (ii) *G has no k-circuit for $k<3$ and* (iii) *there is a 3-gon imbedding $G(M)$.*

Proof. Let n be the chromatic number of H, then $H^0 = \delta_1 \cup \cdots \cup \delta_n$, where δ_1 is an SC-set, $i=1, \cdots, n$, and for any i, $\delta_1 - (\delta_1 \cup \cdots \cup \delta_{i-1} \cup \delta_{i+1} \cup \cdots \cup \delta_n) \neq \phi$. Therefore, we can classify H^0 by such color-classification that

$$H^0 = \gamma_1 \cup \cdots \cup \gamma_n, \qquad \gamma_i \subset \delta_i.$$

If a vertex A is in γ_i, we note γ_i by $\gamma(A)$.

G is constructed by four steps.

(1) We construct H_1 and $H_1(M)$ from H and $H(M)$.

We take the pair of the vertices $A, B \in H^0$ satisfying the following conditions:

(3.3) (i) $\gamma(A) \neq \gamma(B)$ for the color-class defined before.

(ii) A and B are connected in the boundary of a component of $M - H(M)$.

(iii) For a polygonal representation φ_α: $D^2 \to \mathrm{Cl}\,\alpha$, there are such $P_i \in \varphi_\alpha^{-1}(A)$ and $P_j \in \varphi_\alpha^{-1}(B)$ which are not adjoining in $S^1 = Bd\, D^2$.

If there is a pair A, B satisfying (3.3) in H^0, we make an arc (A_1, B_1) joining A_1 and B_1 in α. And we obtain a graph $H_{(A_1, B_1)}$ and an imbedding $H_{(A_1,B_1)}(M)$ such that $H_{(A_1,B_1)}(M) = H(M) \cup (A_1, B_1)$.

By finite repeating the above construction we obtain a graph H_1 and an imbedding $H_1(M)$ satisfying following conditions:

(3.4) (i) $H \subset H_1$, $H^0 = H_1^0$, and $H_1(M)$ is an extension of $H(M)$.

(ii) For a color-classes of H, $\gamma_i (i=1, \cdots, n)$, $\gamma_{1,i} = \gamma_i$ is a color-class of H_1

(iii) H_1 has no 1-cercuit.

(iv) Any component of $M - H_1(M)$ is a 3-gon or 4-gon, and moreover, in case of a 4-gon, all vertices in the boundary are divided to two color-classes which were given in (ii).

(2) We construct a 3-gon imbedding. Let α be a 4-gon which is a component of $M - H_1(M)$. And let P_1, P_2, P_3 and P_4 be the vertices of D^2 related to a polygonal representation φ_α: $D^2 \to \mathrm{Cl}\,\alpha$. Let O be the center of D^2 and (O, P_i) be the radius, (i=1, 2, 3, 4). Note $\varphi_\alpha(O) = A_\alpha$ and $\varphi_\alpha(P_i) = A_{\alpha,i}$. Lastly, let $(A_\alpha, A_{\alpha,i})$ be an arc joining A_α and $A_{\alpha,1}$ which is approximation of $\varphi_\alpha((O, P_i))$ differentiably or piecewise linearly.

Now, we construct a graph H_2, and imbedding $H_2(M)$ and a color-class γ_2 as follows:

(i) $H_2^0 = H_1^0 \cup \{A_\alpha \mid \alpha$ is a 4-gon component of $\mathrm{M} - H_1(M)\}$, $H_2^1 = H_1^1 \cup \{(A_\alpha, A_{\alpha,i}) \mid \alpha$ is a 4-gon component of $M - H_1(M)$, $i=1,2,3,4\}$.

(ii) $H_2(M)$ is made such that $H_2(M) \mid H_1 - H_1(M)$ and $H_2(M)(A_\alpha, A_{\alpha,i}) = (A_\alpha, A_{\alpha,i})$; the arc in M made before.

(iii) As the chromatic number of $H \geqq 3$ and by (iv) of (3.4), there is at least a $\gamma_{1,j}$, for any 4-gon α, such that $\gamma_{1,j} \neq \gamma_1(A_{\alpha,i})$, $i = i, 2, 3, 4$. We take such a $\gamma_{1,j}$ for α, and put it $\gamma_1(\alpha)$. Then $\gamma_{2,j} = \gamma_{1,j} \cup \{A_\alpha \mid \gamma_1(\alpha) = \gamma_{1,j}\}$, $j = 1, \cdots, n$ are color-class of H_2 such that $H_2^0 = \gamma_{2,1} \cup \cdots \cup \gamma_{2,n}$.

H_2, $H_2(M)$, and $\gamma_{2,i}$ constructed above satisfy following conditions:

(3.5) (i) $(H_1 \subset H_2$ and $H_2(M)$ is an extension of $H_1(M)$.

(ii) The color-classification $H_2^0 = \gamma_{2,1} \cup \cdots \cup \gamma_{2,n}$ satisfy $\gamma_{1,i} \subset \gamma_{2,i}$ for $i = 1, \cdots, n$.

(iii) H_2 has no 1-circuit.

(iv) Any component of $M - H_2(M)$ is a 3-gon and the boundary is a 3-circuit.

(3) Next, we take off 2-circuits. If there are 2-circuits in H_2, name one of them D. Let A and B be the vertices of D and $(A, B)_0$ be one of the arcs of D. $(A, B)_0$ is the common boundary of α_1 and α_2 which are components of $M - H_2(M)$. By (iv) of (3.5), there is a vertex C_i of $Bd\,\alpha_i$ which is not A nor B $(i = 1, 2)$.

Now we construct H_D, $H_D(M)$, and $\gamma_{D,i}$ from H_2, $H_2(M)$, and $\gamma_{2,i}$ as follows:

(i) $H_D^0 = H_2^0 \cup \{D_1, E_1, D_2, E_2, F\}$

$H_D^1 = (H_2^1 - (A, B)_0) \cup \{(A_1, D_i), (D_i, E_i), (E_i, B),$
$(C_i, D_i), (C_i, E_i), (F, D_i), (F, E_i) \mid i = 1, 2\} \cup \{(A, F), B, F)\}$.

(ii) $H_D(M)$ is made such that

$$H_0(M) \mid (H_0 - A_1 B)_0) = H_2(M) \mid (H_0 - (A, B)_0),$$

$H_D(M)(F)$ is the center of $H_2(M)((A, B)_0)$ and

$$H_D(M)((A, F) \cup (F, B)) = H_2(M)((A, B)_0),$$

$H_D(M)(D_i)$ and $H_D(M)(E_i)$ is in α_i and the image of (A, D_i), (D_i, E_i), (E_i, B), (C_i, D_i), (C_i, E_i), (F, D_i), and (F, E_i) are the arcs in α_i joining the corresponding points and not intersecting each other, $(i = 1, 2)$.

(iii) $$\gamma_{D,i} = \begin{cases} \gamma_{2,i} \cup \{E_1, E_2\} & \text{if } \gamma_{2,i} = \gamma_2(A). \\ \gamma_{2,i} \cup \{D_1, D_2\} & \text{if } \gamma_{2,i} = \gamma_2(B), \\ \gamma_{2,i} \cup \{F\} & \text{if } \gamma_{2,i} = \gamma_2(C_1), \\ \gamma_{2,i} & \text{if the other case,} \quad i = 1, \cdots, n. \end{cases}$$

$\gamma_{D,i}$ is well defined, for $\gamma_2(A)$, $\gamma_2(B)$, and $\gamma_2(C_1)$ are different each other. This $\gamma_{D,i}$ is the color-class of H_D^0.

By the modification $(H_2, H_2(M), \gamma_2) \rightarrow (H_D, H_D(M), \gamma_D)$, the 2-circuit D is taked away from H_2. Then, by repeating such modifications, we obtain following H_3, $H_3(M)$, and γ_3:

(3.6) (i) $H_2^0 \subset H_3^0$.

(ii) There are color-classes $H_3^0 = \gamma_{3,1} \cup \cdots \cup \gamma_{3,n}$ such that $\gamma_{2,i} \subset \gamma_{3,i}$, $(i = 1, \cdots, n)$.

(iii) H_3 has no k-circuit with $k<3$.

(iv) Any component of $M-H_3(M)$ is a 3-gon and the boundary is a 3-circuit.

(4) $H_3=G$ is the required graph. By (iii) and (iv) of (3.6), G satisfies (ii) and (iii) in the lemma. Then, we show that G also satisfies (i).

By (ii) of (3.5), the chromatic number of H_2 is not larger than n. On the other hand, by (i) of (3.5) and (3.1), the chromatic number of H_2 is not smaller than n. Therefore, H_2 is n-chromatic.

Next, we show that H_3 is n-chromatic.

Let take a subgraph $\tilde{H}_2$ of H_2 which satisfies the following conditions:

(i) $\tilde{H}_2^0=H_2^0$.

(ii) For any two vertices A, B in H_2^0, $(A, B)\in\tilde{H}_2^1$, if and only if $(A, B)\in H_2^1$. Moreover, for $A, B\in H_2^0$, $\tilde{H}_2^1$ contains at most one arc joining A and B.

Namely, $\tilde{H}_2$ is a graph which was made by taking away 2-circuits from H_2.

It can be seen by the method of constructing H_3 and $\tilde{H}_3$ that $\tilde{H}_2\subset H_3$ and the chromatic number of $\tilde{H}_2$ is equal to that of H_2. Then, the chromatic number of $H_3\geqq n$. On the other hand, by (ii) of (3.6) and H_3 being n-chromatic, the chromatic number of $H_3\leqq n$. Therefore, H_3 is n-chromatic. And lemma (3.2) was proved.

By (2.1) and (3.2), we have

Theorem (3.7). *If a graph H has no k-cirquit for $k<3$, the chromatic number of H is $\geqq 3$ and if there exist 2-cell imbedding $H(M)$, then there is a graph G which has the same chromatic number as H and has simplest imbedding into M.*