242. On Products of Quasi-Perfect Maps

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1. Introduction. The purpose of this paper is to give a necessary and sufficient condition for the product map to be a quasi-perfect map under the condition that the image of the product map is a sequential space. Namely we shall prove

Theorem 1. *Let* $f_i: X_i \to Y_i$ $(i=1,2,\cdots)$ *be quasi-perfect maps and* $\prod_{i=1}^{\infty} Y_i$ *a sequential space. Then the following properties are equivalent.*

(a) $\prod_{i=1}^{\infty} f_i$ *is a quasi-perfect map.*

(b) $\prod_{i=1}^{\infty} X_i$ *satisfies the condition* (C_1) *below;*

(C_1): *If each* K_i *is a closed countably compact subset of* X_i, *then* $\prod_{i=1}^{\infty} K_i$ *is countably compact in* $\prod_{i=1}^{\infty} X_i$.

(c) $\prod_{i=1}^{\infty} f_i^{-1}(C_i)$ *is countably compact in* $\prod_{i=1}^{\infty} X_i$ *for each convergent sequence* C_i *in* Y_i. *Here, by a convergent sequence we mean the union of the sequence and its limit point.*

According to S. P. Franklin [4], a space X is called a sequential space if a subset F of X is closed whenever $F \cap C$ is closed for every compact metric subset C of X, and such a space is precisely the quotient of a metric space.

First countable spaces are sequential. Of course, sequential spaces are k-spaces. But the converse does not hold. Indeed, the Stone-Cěch compactification of a normal and non-compact space is not sequential.

We assume all maps are continuous and onto, and all spaces are T_2.

2. Proof of Theorem 1. First of all, we consider the following condition (C_0) on a space X.

(C_0): Each sequence whose closure is countably compact has a subsequence whose closure is compact.

The condition (k_0) in T. Chiba [2] implies the condition (C_0), and the converse is true in regular q-spaces in the sense of E. Michael [7].

A space X is called isocompact by P. Bacon [1] if every closed countably compact is compact.

Lemma 1. *K-spaces, locally isocompact spaces, sequentially compact spaces, and regular spaces whose each point is a* G_δ*-set all satisfy the condition* (C_0).

Proof. Let X be a k-space and $\{x_i; i \in N\}$ a sequence in X whose closure is countably compact. We assume all points x_i are distinct.

Let x_0 be an accumulation point of the set $\{x_i; i \in N\}$. Then $\{x_i; i \in N\} - \{x_0\}$ is not closed. Hence there exists a compact subset K of X such that $K \cap (\{x_i; i \in N\} - \{x_0\})$ is not closed, which implies the infinite subset $K \cap (\{x_i; i \in N\} - \{x_0\})$ has a compact closure. Since a regular countably compact whose each point set is a G_δ-set is first countable, a regular space whose each point is a G_δ-set satisfies the condition (C_0). Since the proof for the other cases is easy, it is omitted.

By the Cantor "diagonal process" (for example, see p. 231 in J. Dugundji [3]), we have

Lemma 2. *If each X_i satisfies the condition* (C_0), *then* $\prod_{i=1}^{\infty} X_i$ *satisfies the condition* (C_1).

Proof of Theorem 1. (a)$\to$(b): Let K_i be closed countably compact in X_i. Then $f_i(K_i)$ is closed countably compact in X_i. Since sequential spaces are k-spaces, from Lemma 1, each Y_i satisfies the condition (C_0). Hence $\prod_{i=1}^{\infty} f_i(K_i)$ is countably compact by Lemma 2. Since $\prod_{i=1}^{\infty} K_i \subset (\prod_{i=1}^{\infty} f_i)^{-1}(\prod_{i=1}^{\infty} f_i(K_i))$ and $(\prod_{i=1}^{\infty} f_i)^{-1}(\prod_{i=1}^{\infty} f_i(K_i))$ is countably compact by the quasi-perfectness of $\prod_{i=1}^{\infty} f_i$, $\prod_{i=1}^{\infty} K_i$ is countably compact.

(b)$\to$(c): Since each convergent sequence C_i in Y_i is compact and f_i is quasi-perfect, $f_i^{-1}(C_i)$ is closed countably compact. Hence $\prod_{i=1}^{\infty} f_i^{-1}(C_i)$ is countably compact for each convergent sequence C_i in Y_i by (b).

(c)$\to$(a): Let F be a closed subset of $\prod_{i=1}^{\infty} X_i$. We shall prove $(\prod_{i=1}^{\infty} f_i)(F)$ is closed. Suppose $(\prod_{i=1}^{\infty} f_i)(F)$ is not closed. Since $\prod_{i=1}^{\infty} Y_i$ is sequential, from S. P. Franklin [4; Proposition 1.1], there exists a sequence $\{y_n; n \in N\}$ such that $y_n = (y_{in}) \in (\prod_{i=1}^{\infty} f_i)(F)$, that is, $\prod_{i=1}^{\infty} (f_i^{-1}(y_{in})) \cap F \neq \phi$, and $\{y_n; n \in N\}$ converges to a point $p = (p_i) \notin (\prod_{i=1}^{\infty} f_i)(F)$. Let $A_{in} = \{y_{ij}; j \geqq n\} \cup \{p_i\}$, and $F_n = \prod_{i=1}^{\infty} (f_i^{-1}(A_{in})) \cap F$. Then each sequence A_{in} is a convergent sequence. Hence $\prod_{i=1}^{\infty} (f_i^{-1}(A_{in}))$ is closed countably compact by (c). Moreover $F_n \supset F_{n+1} \neq \phi$ for each $n \in N$. Therefore $\bigcap_{n=1}^{\infty} F_n \neq \phi$, which implies $\prod_{i=1}^{\infty} (f_i^{-1}(p_i)) \cap F \neq \phi$. But this is impossible. Thus $\prod_{i=1}^{\infty} f_i$ is closed map. Since $(\prod_{i=1}^{\infty} f_i)^{-1}(y)$ is countably compact for each point y of $\prod_{i=1}^{\infty} Y_i$, $\prod_{i=1}^{\infty} f_i$ is a quasi-k-space.

Remark. (i): From the above proof, we see that a map of X onto a sequential space Y is quasi-perfect if and only if $f^{-1}(C)$ is countably compact for each convergent sequence C in Y. From this and S. Hanai [5; Corollary 1.7], a space X is countably compact if and only if the projection $P: X \times Y \to Y$ is closed for every sequential space Y.

(ii): The product map of quasi-perfect maps need not be closed and not even be quotient.

As for the former, let $f: [0, \Omega) \rightarrow \{0\}$ and i be the identity of $[0, \Omega]$, where Ω is the first uncountable ordinal number. Then $f \times i$ is not closed. As for the latter, let X be the product of $[0, \Omega)$ with itself, and Y be the space obtained from X by identifying all points of the diagonal set of X, and let f be the quotient map of X onto Y. Then f is quasi-perfect, but it is not a bi-quotient map defined by E. Michael [8]. From E. Michael [8; Theorem 1.3], $f \times i_Z$ is not quotient for some paracompact space Z.

3. Applications of Theorem 1. K. Morita [9] has introduced the notion of M-spaces and proved a space X is an M-space if and only if there exist a metric space Y and a quasi-perfect map of X onto Y. Then, from Theorem 1, we have

Corollary 1. *Let $\prod_{i=1}^{\infty} X_i$ satisfy the condition* (C_1), *and each X_i be an M-space. Then $\prod_{i=1}^{\infty} X_i$ is an M-space.*

From Theorem 1 and Lemma 2, we have

Corollary 2. *Let $f_i: X_i \rightarrow Y_i$ $(i=1, 2, \cdots)$ be quasi-perfect maps. If each X_i satisfies the condition (C_0), and Y_i is first countable for each $i \in N$, then $\prod_{i=1}^{\infty} f_i$ is a quasi-perfect map.*

Corollary 3. *Let $f_i: X_i \rightarrow Y_i$ $(i=1, 2, \cdots)$ be closed, X_i normal, and $\prod_{i=1}^{\infty} X_i$ sequential. Then $\prod_{i=1}^{\infty} f_i$ is quotient if and only if $\prod_{i=1}^{\infty} Y_i$ is sequential.*

Proof. The "only if" part is obvious, for the quotient of a sequential space is sequential.

"if"; Since $\prod_{i=1}^{\infty} Y_i$ is sequential and $C \subset \prod_{i=1}^{\infty} P_i(C)$ for every subset C of $\prod_{i=1}^{\infty} Y_i$, where P_i is the projection of $\prod_{i=1}^{\infty} Y_i$ onto Y_i, a subset F of $\prod_{i=1}^{\infty} Y_i$ is closed whenever $F \cap \prod_{i=1}^{\infty} C_i$ is closed for every subset $\prod_{i=1}^{\infty} C_i$ of $\prod_{i=1}^{\infty} Y_i$, where each C_i is a compact metric subset of Y_i. Hence, $\prod_{i=1}^{\infty} f_i$ is quotient whenever $\prod_{i=1}^{\infty} (f_i | f_i^{-1}(C_i))$ is quotient for every compact metric subset C_i of Y_i. We can then assume Y_i are first countable. From K. Morita and S. Hanai [9], there exists $g_i: A_i \rightarrow Y_i$ such that g_i is quasi-perfect, A_i is a closed subset of X_i. Since $\prod_{i=1}^{\infty} A_i$ satisfies the condition (C_1), from Theorem 1, $\prod_{i=1}^{\infty} g_i = \prod_{i=1}^{\infty} f_i \cdot \prod_{i=1}^{\infty} i_{A_i}$ is quotient, which implies that $\prod_{i=1}^{\infty} f_i$ is quotient.

Remark. Even if $Y_1 \times Y_2$ is separable metric, $f_1 \times f_2$ need not be closed. Indeed, let $f: [0, 1) \rightarrow \{0\}$ and i be the identity of $[0, 1]$. Then $f \times i$ is not closed.

Now, we consider the product of closed map.

Lemma 3. *Let $f_i: X_i \rightarrow Y_i$ $(i=1, 2)$ be closed maps, Y_i nondiscrete, sequential spaces, and $f_1 \times f_2$ closed. Then f_1 and f_2 are quasi-perfect maps.*

Proof. Since Y_1 is a non-discrete sequential space, there exists a point y_1 which, considered as a subset, is not open in some metric subset C of Y_1. As $f_1\times f_2$ is closed, $(f_1|f_1^{-1}(C))\times f_2$ is also closed. Since $f_1^{-1}(y_1)$ is a G_δ-set of $f_1^{-1}(C)$, from the proof given by T. Ishii [6; Theorem 2.3], we see that if we suppose $f_2^{-1}(y_2)$ is not countably compact for some point y_2 of Y_2, then $(f_1|f_1^{-1}(C))\times f_2$ is not closed. This is impossible. Therefore $f_2^{-1}(y_2)$ is countably compact for each point y_2 of Y_2. Since f_2 is closed, f_2 is a quasi-perfect map. Similarly, f_1 is a quasi-perfect map.

From Theorem 1 and Lemma 3, we have

Theorem 2. *Let $f_i: X\to Y_i$ $(i=1,2,\cdots)$ be closed maps, Y_i non-discrete spaces, $\prod_{i=1}^\infty Y_i$ a sequential space, and $\prod_{i=1}^\infty X_i$ satisfy the condition* (C_1). *Then the following properties are equivalent.*

(a) *$\prod_{i=1}^\infty f_i$ is a closed map.*

(b) *$\prod_{i=1}^\infty f_i$ is a quasi-perfect map.*

(c) *Each f_i is a quasi-perfect map.*

Theorem 3. *Let $f_i: X_i\to Y_i$ $(i=1,2)$ be closed maps, X_i sequential M-spaces, and Y_i non-discrete, normal spaces. Then $f_1\times f_2$ is closed if and only if f_1 and f_2 are quasi-perfect maps.*

Proof. The "only if" part follows from Lemma 3.

"if"; From K. Morita [11; Theorem 2.2], Y_i are M-spaces. While, Y_i are sequential. Hence $Y_1\times Y_2$ is sequential by [12; Corollary 2.3]. Consequently, $f_1\times f_2$ is a quasi-perfect map by Theorem 1.

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