183. On Weakly Compact Spaces

By Masao SAKAI

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1971)

A topological space S is said to be *AU-weakly compact*, if every countable open covering of S contains a finite subfamily whose union is deuse in S, and S is said to be *MP-weakly compact*, if every pairwise disjoint infinite family of open sets O_α, $\alpha \in A$, has a point $p \in S$ whose every neighbourhood meets infinitely many O_α. The point p is called a *cluster point* of the family $\{O_\alpha\}_{\alpha \in A}$. K. Iseki [1] [2] [3] and S. Kasahara [4] proved the following:

Proposition. *The following properties of a regular space S are equivalent:*

(1) *S is AU-weakly compact.*

(2) *S is MP-weakly compact.*

(3) *Every locally finite family of open sets O_α contains a finite subfamily whose union covers the union of all O_α.*

(4) *Every locally finite open covering of S contains a finite subcovering.*

We shall prove only that (2)$\rightarrow$(3) using the following:

Lemma. *Every point-finite covering of a topological space contains an irreducible subcovering.*

This lemma was proved by R. Arens and J. Dugundji [5].

Proof that (2)$\rightarrow$(3). Let S be a regular *MP*-weakly compact space and let $\{O_\alpha\}_{\alpha \in A}$ be a locally finite family of open sets of S. By the lemma, there is an irreducible subfamily $\{O_\beta\}_{\beta \in B}$ such that $\bigcup_{\beta \in B} O_\beta = \bigcup_{\alpha \in A} O_\alpha$. We shall prove that B is a finite set. Let us assume that B is an infinite set. By the irreducibility of $\{O_\beta\}_{\beta \in B}$ for every $\beta \in B$, $O_\beta - \bigcup_{\gamma \in B-\{\beta\}} O_\gamma$ is non-empty, then it contains a point p_β such that $p_\beta \in O_\beta$ and $p_\beta \bar{\in} O_\gamma$, $\gamma \in B-\{\beta\}$. By the regularity of the space S, every p_β has an open neighbourhood V_β such that $\bar{V}_\beta \subset O_\beta$. It is easily seen that for every $\beta \in B$ $p_\beta \in V_\beta$ and $p_\beta \bar{\in} \bar{V}_\gamma$, $\gamma \in B-\{\beta\}$. By the locally finiteness of $\{O_\beta\}_{\beta \in B}$, $\bigcup_{\gamma \in B-\{\beta\}} \bar{V}_\gamma$ is closed, then $W_\beta = V_\beta - \bigcup_{\gamma \in B-\{\beta\}} \bar{V}_\gamma$ is open and contains p_β. It is obvious that the open infinite family $\{W_\beta\}_{\beta \in B}$ is pairwise disjoint and locally finite. By the property (2), the family $\{W_\beta\}_{\beta \in B}$ has at least one cluster point, contrary to the locally finiteness of the family $\{W_\beta\}_{\beta \in B}$. Then B must be a finite set and the proof of (2)$\rightarrow$(3) is completed.

Let S be a topological space. Each family of regularly closed sets $\bar{O}_\alpha$, $\alpha \in A$, of S is called a *regularly closed family*, and each covering of S

composed of regularly closed sets $\bar{O}_\alpha$, $\alpha \in A$, a *regularly closed covering.*

Theorem. *The following properties of a topological space S are equivalent.*

(1) *S is AU-weakly compact.*

(2) *Every countable non-empty open family $\{O_n\}_{n=1}^\infty$ having the finite intersection property has the non-empty intersection $\bigcap_{n=1}^\infty \bar{O}_n$.*

(3) *S is MP-weakly compact.*

(4) *Every locally finite family of regularly closed sets $\bar{O}_\alpha$ of S contains a finite subfamily whose union covers the union of all $\bar{O}_\alpha$.*

(5) *Every locally finite regularly closed covering of S contains a finite subcovering.*

Proof. In (5), "*covering*" may be replaced by "*countable covering*" and "*a finite subcovering*" by "*a proper subcovering*". We shall prove that (1)→(2)→(3)→(4)→(5)→(3)→(1). K. Iséki [3] proved that (1)⇄(2) and (3)→(1), K. Iséki [1] proved that (2)→(3) in topological spaces. It is obvious that (4)→(5). We must prove that (3)→(4) and (5)→(3).

Proof that (3)→(4). Let S be a topological space and let $\{\bar{O}_\alpha\}_{\alpha \in A}$ be a regularly closed family. Put $S_1 = \bigcup_{\alpha \in A} \bar{O}_\alpha$. In virtue of the lemma, there is an irreducible subfamily $\{\bar{O}_\beta\}_{\beta \in B}$ such that $S_1 = \bigcup_{\beta \in B} \bar{O}_\beta$. We shall prove that B is a finite set. Let us assume that B is an infinite set. For every $\beta \in B$, put $W_\beta = O_\beta - \bigcup_{\gamma \in B - \{\beta\}} \bar{O}_\gamma$. If $W_\beta = \phi$ for some $\beta \in B$, $O_\beta \subset \bigcup_{\gamma \in B-\{\beta\}} \bar{O}_\gamma$. By the locally finiteness of $\{\bar{O}_\alpha\}_{\alpha \in A}$ the set $\bigcup_{\gamma \in B-\{\beta\}} \bar{O}_\gamma$ is closed, then $\bar{O}_\beta \subset \bigcup_{\gamma \in B-\{\beta\}} \bar{O}_\gamma$ contrary to the irreducibility of $\{\bar{O}_\beta\}_{\beta \in B}$. Therefore $W_\beta \neq \phi$ for any $\beta \in B$. By $W_\beta \subset O_\beta$, $\{W_\beta\}_{\beta \in B}$ is locally finite. It is obvious that $\{W_\beta\}_{\beta \in B}$ is a pairwise disjoint open infinite family. By the property (3), $\{W_\beta\}_{\beta \in B}$ has at least one cluster point, contrary to the locally finiteness of $\{W_\beta\}_{\beta \in B}$. Therefore, B must be a finite set. The proof of (3)→(4) is completed.

Proof that (5)→(3). Let S be a topological space which does not satisfy the property (3). Then there is a pairwise disjoint open infinite family $\{O_n\}_{n=1}^\infty$ which has no cluster point. Therefore, the family $\{O_n\}_{n=1}^\infty$ is locally finite. If $\bigcup_{n=1}^\infty \bar{O}_n = S$, the family $\{\bar{O}_n\}_{n=1}^\infty$ is a locally finite regularly closed infinite covering which has no proper subcovering. If $\bigcup_{n=1}^\infty \bar{O}_n \neq S$, by the locally finiteness of $\{\bar{O}_n\}_{n=1}^\infty$, $S - \bigcup_{n=1}^\infty \bar{O}$ is non-empty open. Then the family $\{\overline{S - \bigcup_{n=1}^\infty \bar{O}_n}, \bar{O}_1, \bar{O}_2, \cdots, \bar{O}_n, \cdots\}$ is a locally finite regularly closed covering which has no proper subcovering. Thus, the proof of (5)→(3) is completed.

References

[1] K. Iséki: On weakly compact regular spaces. I. Proc. Japan Acad., **33**, 252–254 (1957).

[2] ——: AU-covering theorem and compactness. Proc. Japan Acad., **33**, 363–367 (1957).

[3] ——: Generalisations of the notion of compactness. Rev. Japan Pures Appl., **6**, 31–63 (1961).

[4] S. Kasahara: On weakly compact regular spaces. II. Proc. Japan Acad., **33**, 255–259 (1957).

[5] R. Arens and J. Dugundji: Remark on the concept of compactness. Portugaliae Math., **9**, 141–143 (1950).