182. A Sharp Form of Gårding's Inequality for a Class of Pseudo-Differential Operators

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(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1971)

Introduction.

The present paper is concerned with an algebra of a class of pseudo-differential operators and a sharp form of Gårding's inequality (see [3], [4], [5], and [7]), which are important for the study of the Cauchy problem for pseudo-differential equations of parabolic type. Let $\lambda(\xi)$ be a basic weight function which means that $\lambda(\xi)$ is a real valued C^∞-function satisfying $\lambda(\xi)\geqq 1$ and $|\partial_\xi^\alpha \lambda(\xi)|\leqq C_\alpha \lambda(\xi)^{m-|\alpha|}$ (see [5]).

Then we say $p(x,\xi)\in S^m_{0,\lambda}$ when $p(x,\xi)\lambda(\xi)^{-m}\in \mathscr{B}(R^{2n}_{(x,\xi)})$.[1)]

For $p(x,\xi)\in S^m_{0,\lambda}$, the pseudo-differential operator P is defined by

$$Pu(x)=p(X,D_x)u(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\!\!{}^-\xi$$

where $d\!\!{}^-\xi=(2\pi)^{-n}d\xi$ and $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$.

From Theorem 1.1, we have

(1) *Let* $p(x,\xi)\in S^m_{0,\lambda}$. *Then there exists* $p^*(x,\xi)\in S^m_{0,\lambda}$ *such that*

$$(p(X,D_x)u,v)=(u,p^*(X,D_x)v) \qquad \text{for any } u,v\in\mathcal{S}.$$

(2) *Let* $p_j(x,\xi)\in S^{m_j}_{0,\lambda}$ $(j=1,2)$. *Then there exists* $p_{1,2}(x,\xi)\in S^{m_1+m_2}_{0,\lambda}$ *such that*

$$p_{1,2}(X,D_x)u=p_1(X,D_x)\cdot p_2(X,D_x)u \qquad \text{for any } u\in\mathcal{S}.$$

These properties mean that the operator class corresponding to $S^\infty_{0,\lambda}=\bigcup_{-\infty<m<\infty}S^m_{0,\lambda}$ forms an algebra.

Recently Calderón and Vaillancourt [1] proved the L^2-boundedness $\|Pu\|_{L^2(R^n)}\leqq C\|u\|_{L^2(R^n)}$ for $p(x,\xi)\in S^0_{0,\lambda}=\mathscr{B}(R^{2n}_{(x,\xi)})$. Using this estimate essentially, we obtain the inequality

$$\|Pu\|_s\leqq C\|u\|_{s+m} \qquad \text{for } p(x,\xi)\in S^m_{0,\lambda}$$

where $\|u\|_s^2=\|u\|_{s,\lambda}^2=\int \lambda(\xi)^{2s}|\hat{u}(\xi)|^2 d\!\!{}^-\xi$ (see Corollary 1.2).

From this inequality and the Friedrichs' approximation we can derive a sharp form of Gårding's inequality

$$\mathcal{R}e(p(X,D_x)u,u)\geqq -C\|u\|^2_{1/2(m-1)}$$

when $p(x,\xi)\geqq 0$ and $\partial_\xi^\alpha p(x,\xi)\in S^{m-|\alpha|}_{0,\lambda}$ for $|\alpha|\leqq 2$ (Theorem 2.3 (2)).

The proofs of our results are based on the method in Kumano-go [4].

1) $\mathscr{B}(R^N)=\{u\in C^\infty(R^N);\ |\partial_x^\alpha u(x)|\leqq C_\alpha \text{ for any } \alpha\}$.

I wish to thank Professor H. Kumano-go for a number of helpful conversations.

1. The properties of pseudo-differential operators.

Lemma 1.1 (Calderón and Vaillancourt [1]). *If* $p(x,\xi)\in\mathscr{B}(R^{2n}_{(x,\xi)})$, *then* $\|Pu\|_{L^2(R^n)}\leqq C\|u\|_{L^2(R^n)}$ *hold for* $u\in\mathcal{S}$.

Lemma 1.2 (Kumano-go [5]). *Let* $\lambda(\xi)$ *be a basic weight function. Then we have,*

(1.1) $\lambda(\xi+\eta)^m\leqq C_m\lambda(\xi)^m\langle\eta\rangle^{|m|}$ [2]

(1.2) $C^{-1}\lambda(\xi)\leqq\lambda(\xi+\lambda(\xi)^{1/2}\sigma)\leqq C\lambda(\xi)$ for $|\sigma|\leqq 1$.

Theorem 1.1. *Let* $\lambda(\xi)$ *and* $\lambda'(\xi)$ *be basic weight functions. If* $p(x,\xi,x',\xi')\lambda(\xi)^{-m}\lambda'(\xi')^{-m'}\in\mathscr{B}(R^{4n}_{(x,\xi,x',\xi')})$ *and*

$$Pu(x)=\iiint e^{i(x\cdot\xi-\xi\cdot x'+x'\cdot\xi')}\,p(x,\xi,x',\xi')\hat{u}(\xi')d\xi'\cdot dx'\cdot d\xi,$$

and if we put $p_L(x,\xi)=\iint e^{-iz\cdot\zeta}\langle z\rangle^{-n_0}\langle D_z\rangle^{n_0}p(x,\xi+\zeta,x+z,\xi)dz\cdot d\zeta$, *where* n_0 *is an even integer larger than* n.

Then,

(1.3) $p_L(x,\xi)\lambda(\xi)^{-m}\lambda'(\xi)^{-m'}\in\mathscr{B}(R^{2n}_{(x,\xi)})$,

(1.4) $Pu(x)=p_L(X,D_x)u(x)$.

We can prove the theorem, using (1.1).

Corollary 1.1. (1) *Let* $p(x,\xi)\in S^m_{0,\lambda}$, *and let*

$$p^*(x,\xi)=\iint e^{-iz\cdot\zeta}\langle z\rangle^{-n_0}\langle D_z\rangle^{n_0}\overline{p(x+z,\xi+\zeta)}dz\cdot d\zeta,$$

then we get $p^*(x,\xi)\in S^m_{0,\lambda}$ *and*

$$(p(X,D_x)u,v)=(u,p^*(X,D_x)v)\qquad\text{for } u,v\in\mathcal{S}.$$

(2) *Let* $p_j(x,\xi)\in S^{m_j}_{0,\lambda_j}(j=1,2)$. *Then there exists* $p_{1,2}(x,\xi)$ *such that* $p_{1,2}(X,D_x)u=p_1(X,D_x)\cdot p_2(X,D_x)u$ *and* $p_{1,2}(x,\xi)\lambda_1(\xi)^{-m_1}\lambda_2(\xi)^{-m_2}\in\mathscr{B}(R^{2n}_{(x,\xi)})$.

Corollary 1.2. *Let* $p(x,\xi)\in S^m_{0,\lambda}$. *Then we have,* $\|p(X,D_x)u\|_s\leqq C\|u\|_{s+m}$ *for* $u\in\mathcal{S}$.

Theorem 1.2. *Let* $\partial^\alpha_\xi p(x,\xi,x',\xi')\lambda(\xi)^{-m+\rho(|\alpha|)}\cdot\lambda'(\xi')^{-m'}\in\mathscr{B}(R^{4n})$, *and* $p_\alpha(x,\xi)=(-i\partial_{x'})^\alpha\partial^\alpha_\xi p(x,\xi,x',\xi')\big|_{\substack{x'=x\\ \xi'=\xi}}$ *where* $\rho(0)=0$ *and* $\rho(j)\leqq\rho(j+1)$. *Then we get,*

$$\Big\{p_L(x,\xi)-\sum_{|\alpha|<N}\frac{1}{\alpha!}p_\alpha(x,\xi)\Big\}\lambda(\xi)^{-m+\rho(N)}\lambda'(\xi)^{-m'}\in\mathscr{B}(R^{2n}).$$

Corollary 1.3. *Let* $\partial^\alpha_\xi p(x,\xi)\in S^{m-\rho(|\alpha|)}_{0,\lambda}$ *and*

$$p^*_\alpha(x,\xi)=(-i\partial_x)^\alpha\partial^\alpha_\xi p(x,\xi),$$

then

$$p^*(x,\xi)-\sum_{|\alpha|<N}\frac{1}{\alpha!}p^*_\alpha(x,\xi)\in S^{m-\rho(N)}_{0,\lambda}.$$

Corollary 1.4. *Let* $p(x,\xi)\in S^m_{0,\lambda}$. *Then,*

$$p_{L,s}(X,D_x)=\lambda(D_x)^s\cdot p(X,D_x)$$

and

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2) $\langle\eta\rangle=(1+|\eta|^2)^{1/2}$.

$$p_{L,s}(x,\xi)-\sum_{|\alpha|<N}\frac{1}{\alpha!}p_{L,s,(\alpha)}(x,\xi)\in S_{0,\lambda}^{m+s-N}$$

where $p_{L,s,(\alpha)}(x,\xi)=\partial_\xi^\alpha\{\lambda(\xi)^s\}\cdot(-i\partial_x)^\alpha p(x,\xi)$.

We can prove Theorem 1.2 by using the Taylor expansion of $p(x,\xi+\zeta,x+z,\xi)$ in ζ at $\zeta=0$.

2. Friedrichs' approximation and a sharp form of Gårding's inequality.

Let $q(\sigma)\in C_0^\infty(R^n)$ be an even function satisfying $\int q(\sigma)^2 d\sigma=1$ and supp $q\subset\{\sigma;|\sigma|\leqq 1\}$ and $F(\xi,\zeta)=\lambda(\xi)^{-n/4}\,q((\zeta-\xi)\lambda(\xi)^{-1/2})$.

We can easily prove by induction the following

Lemma 2.1 (Kumano-go [4]). *There exist functions*

$$\psi_{\beta,\gamma,\gamma_1}(\xi)\in S_{0,\lambda}^{-|\beta|+|\gamma-\gamma_1/2|}$$

such that

$$\partial_\xi^\beta F(\xi,\zeta)=\lambda(\xi)^{-n/4}\sum_{\substack{|\gamma|\leqq|\beta|\\ \gamma_1\leqq\gamma}}\psi_{\beta,\gamma,\gamma_1}(\xi)((\xi-\zeta)\lambda(\xi)^{-1/2})^{\gamma_1}$$

$$\partial_\sigma^\gamma q((\zeta-\xi)\lambda(\xi)^{-1/2})\quad\text{and}\quad\partial_\xi^\alpha\psi_{\beta,\gamma,\gamma_1}(\xi)\in S_{0,\lambda}^{-|\beta|+|\gamma-\gamma_1|/2-|\alpha|}.$$

Theorem 2.1. *For* $p(x,\xi)\in S_{0,\lambda}^m$ *we put*

$$p_F(\xi,x',\xi')=\int F(\xi,\zeta)p(x',\zeta)F(\xi',\zeta)d\zeta.$$

Then $p_F(\xi,x',\xi')$ *satisfies the following properties*:

(2.1) $\partial_\xi^\beta\partial_{\xi'}^{\beta'}p_F(\xi,x,\xi')\lambda(\xi)^{-m+|\beta|/2}\lambda(\xi')^{|\beta'|/2}\in\mathscr{B}$,

(2.2) $p_{F,L}(x,\xi)\in S_{0,\lambda}^m$,

(2.3) *If* $\partial_{\xi_j}p(x,\xi)\in S_{0,\lambda}^{m-1}$ $(j=1,\cdots,n)$, *then* $p_{F,L}(x,\xi)-p(x,\xi)\in S_{0,\lambda}^{m-1/2}$,

(2.4) *If* $\partial_\xi^\alpha p(x,\xi)\in S_{0,\lambda}^{m-|\alpha|}$ for $|\alpha|\leqq 2$, *then* $p_{F,L}(x,\xi)-p(x,\xi)\in S_{0,\lambda}^{m-1}$,

(2.5) *If* $p(x,\xi)\geqq 0$, *then* $(p_F u,v)=(u,p_F v)$ *and* $(p_F u,u)\geqq 0$,

(2.6) $|\mathscr{R}e(p_F u,u)|\leqq|\mathscr{R}e\,p|_{0,m}\|u\|_{m/2}^2+C\|u\|_{1/2(m-1)}^2$

where $|\mathscr{R}e\,p|_{0,m}=\sup_{(x,\xi)}\{|\mathscr{R}e\,p(x,\xi)|\lambda(\xi)^{-m}\}$.

A sketch of the proof. Using Lemma 2.1 and (1.2), we can prove (2.1), and (2.2) follows from (2.1) and Theorem 1.1.

By Theorem 1.2, $p_{F,L}(x,\xi)-p_{F,0}(x,\xi)\in S_{0,\lambda}^{m-1/2}$ and

$$\begin{aligned}p_{F,0}(x,\xi)&=\int F(\xi,\zeta)^2\,p(x,\zeta)d\zeta\\&=\int q(\sigma)^2\,p(x,\xi+\lambda(\xi)^{1/2}\,\sigma)d\sigma\\&=p(x,\xi)+\sum_{j=1}^n\int q(\sigma)^2\int_0^1 p^{(j)}(x,\xi+\lambda(\xi)^{1/2}\,t\sigma)dtd\sigma\,\lambda(\xi)^{1/2}\end{aligned}$$

where $p^{(j)}(x,\xi)=\partial_{\xi_j}p(x,\xi)$.

By the assumption we can prove

$$\int q(\sigma)^2\int_0^1 p^{(j)}(x,\xi+\lambda(\xi)^{1/2}\,t\sigma)dtd\sigma\in S_{0,\lambda}^{m-1}.$$

Hence we obtain (2.3).

By Theorem 1.2 and the assumption we have

$$p_{F,L}(x,\xi)-p_{F,0}(x,\xi)-\sum_{j=1}^{n} p_{F,j}(x,\xi)\in S_{0,\lambda}^{m-1}.$$

Then using the assumptions of $q(\sigma)$ we get,

$$p_{F,0}(x,\xi)-p(x,\xi)\in S_{0,\lambda}^{m-1} \quad \text{and} \quad p_{F,j}(x,\xi)\in S_{0,\lambda}^{m-1}.$$

Thus we obtain (2.4).

The inequality (2.5) can be shown easily.

Using (2.5) for the operator defined by $p^{(\pm)}(x,\xi)=|\mathcal{R}e p|_{0,m}\lambda(\xi)^m \pm p(x,\xi)(\geqq 0)$, we can prove (2.6).

Lemma 2.2. (1) *Suppose that* $\partial_\xi^\alpha p(x,\xi)\in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha|\leqq 1$. *Then we have,*

$$|\mathcal{R}e(Pu,u)|\leqq |p|_{0,m}\,\|u\|_{m/2}^2+C\,\|u\|_{1/2(m-1/2)}^2.$$

(2) *Suppose that* $\partial_\xi^\alpha p(x,\xi)\ S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha|\leqq 2$. *Then we have,*

$$|\mathcal{R}e(Pu,u)|\leqq |p|_{0,m}\,\|u\|_{m/2}^2+C\|u\|_{1/2(m-1)}^2.$$

A sketch of the proof.

$$\begin{aligned}|\mathcal{R}e(p(X,D_x)u,u)&\leqq|\mathcal{R}e(p_F u,u)|+|\mathcal{R}e((P_F-p(X,D_x)u,u)|\\ &\leqq|\mathcal{R}e p|_{0,m}\,\|u\|_{m/2}^2+C\,\|u\|_{1/2(m-1)}^2\\ &\quad+|((p_{F,L}(X,D_x)-p(X,D_x))u,u)|.\end{aligned}$$

If we assume that $\partial_{\xi_j}p(x,\xi)\in S_{0,\lambda}^{m-1}$, then by (2.3) $p_{F,L}(x,\xi)-p(x,\xi)\in S_{0,\lambda}^{m-1/2}$. Hence by Corollary 1.2,

$$|((p_{F,L}-p)u,u)|\leqq\|(p_{F,L}-p)u\|_{-1/2(m-1/2)}\|u\|_{1/2(m-1/2)}\leqq C\,\|u\|_{1/2(m-1/2)}^2.$$

Thus we obtain (1), and (2) can be shown by the same way.

Theorem 2.2. (1) *Suppose that* $\partial_\xi^\alpha p(x,\xi)\in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha|\leqq 1$. *Then we get,*

$$\|p(X,D_x)u\|_s^2\leqq|p|_{0,m}^2\,\|u\|_{s+m}^2+C\,\|u\|_{s+m-1/4}^2.$$

(2) *Suppose that* $\partial_\xi^\alpha p(x,\xi)\in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha|\leqq 2$. *Then we get,*

$$\|p(X,D_x)u\|_s^2\leqq|p|_{0,m}^2\,\|u\|_{s+m}^2+C\,\|u\|_{s+m-1/2}^2.$$

A sketch of the proof. By Corollary 1.1 and Theorem 1.2,

$$p^*(X,D_x)\cdot\lambda(D_x)^{2s}\cdot p(X,D_x)u=p_s(X,D_x)u+p_{s,1}(X,D_x)u$$

where $p_s(x,\xi)=\lambda(\xi)^{2s}\ |p(x,\xi)|^2$ and $p_{s,1}(x,\xi)\in S_{0,\lambda}^{(2m+s)-1}$. Hence, using Lemma 2.2, we obtain the theorem.

Theorem 2.3. *Let* $p(x,\xi)\in S_{0,\lambda}^m$ *satisfy that* $p(x,\xi)\geqq\delta\lambda(\xi)^m(\delta\geqq 0)$. *Then we get,*

(1) *If* $\partial_\xi^\alpha p(x,\xi)\in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha|\leqq 1$,

$$\mathcal{R}e(p(X,D_x)u,u)\geqq\delta\|u\|_{m/2}^2-C\,\|u\|_{1/2(m-1/2)'}^2.$$

(2) *If* $\partial_\xi^\alpha p(x,\xi)\in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha|\leqq 2$,

$$\mathcal{R}e(p(X,D_x)u,u)\geqq\delta\,\|u\|_{m/2}^2-C\,\|u\|_{1/2(m-1)}^2.$$

A sketch of the proof. Let $p_0(x,\xi)=p(x,\xi)-\delta\lambda(\xi)^m$. By Theorem 2.1 (2.5), $(p_{0,F}u,u)\geqq 0$. Hence,

$$\mathcal{R}e(p_0(X,D_x)u,u)\geqq\mathcal{R}e((p_0-p_{0,F})u,u)=\mathcal{R}e((p_0-p_{0,F,L})u,u).$$

Thus using (2.3) or (2.4), we obtain the theorem.

Theorem 2.4. *Let* $p(x,\xi)\in S_{0,\lambda}^m$ *satisfy that* $p(x,\xi)\geqq\delta\cdot\lambda(\xi)^m$.

(1) *If* $\partial_\xi^\alpha p(x, \xi) \in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha| \leqq 1$, *then*

$$\delta^2 \|u\|_{s+m}^2 \leqq \|p(X, D_x)u\|_s^2 + C\|u\|_{s+m-1/4}^2.$$

(2) *If* $\partial_\xi^\alpha p(x, \xi) \in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha| \leqq 2$, *then*

$$\delta^2 \|u\|_{s+m}^2 \leqq \|p(X, D_x)u\|_s^2 + C\|u\|_{s+m-1/2}^2.$$

Theorem 2.4 can be proved by using Theorem 2.3.

Theorem 2.3 and Theorem 2.4 are important for studying the Cauchy problem for a parabolic pseudo-differential equation.

3. An application for a pseudo-differential equation of parabolic type.

In section 3 we state the fundamental theorem which is useful for proving the existence theorem of the Cauchy problem for a pseudo-differential equation of parabolic type.

S. Kaplan [6] proved the existence theorem of the Cauchy problem for a parabolic differential equation. Using the results in sections 1, and 2, we can get the similar results to [6] for the case of pseudo-differential equation.

Definition 3.1. $H_{r,s} = u \in \mathcal{S}'(R_{(t,x)}^{n+1})$;

$$\lambda_1(\tau, \xi)^r \lambda(\xi)^s \hat{u}(\tau, \xi) \in L^2(R_{(\tau,\xi)}^{n+1})$$

where $\lambda_1(\tau, \xi) = (\tau^2 + \lambda(\xi)^{2m})^{1/2m}$, $m \geqq 1$.

(In connection with $H_{r,s}$, see Hörmander [2].)

Let $p_j(t, x, \xi) \in S_{0,\lambda}^m$ $(j=1, 2)$ be real valued symbols, and $L = \partial_t + p_1(t, X, D_x) + \sqrt{-1} \cdot p_2(t, X, D_x)$.

Theorem 3.1. *If* $\partial_\xi^\alpha p_j(t, x, \xi) \in S_{0,\lambda}^{m-|\alpha|}$ *for* $|\alpha| \leqq 1$, *and* $p_1(t, x, \xi) \geqq \delta\lambda(\xi)^m$ $(\delta > 0)$, *there exist* $\eta_0 > 0$ *such that for any* $\eta > \eta_0$,

$$C^{-1}\|u\|_{r,s} \leqq \|(L+\eta)u\|_{r-m,s} \leqq C\|u\|_{r,s} \tag{3.1}$$

$$C^{-1}\|u\|_{r,s} \leqq \|(L^*+\eta)u\|_{r-m,s} \leqq C\|u\|_{r,s} \tag{3.2}$$

where $C = C(r, s, m, \eta)$.

Corollary 3.1. *Under the same conditions in Theorem* 3.1, $L+\eta$ *is an onto, one to one and bounded mapping from* $H_{r,s}$ *to* $H_{r-m,s}$.

Theorem 3.1 can be proved by using Corollaries 1.3, 1.4, Theorems 2.3, 2.4 (1) and

$$|\sqrt{-1}\,\tau + p_1(t, x, \xi) + \sqrt{-1}\,p_2(t, x, \xi)| \geqq c_0(\tau^2 + \lambda(\xi)^{2m})^{1/2}.$$

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