227. Remark on the Essential Spectrum of Symmetrizable Operators

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(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1971)

1. Introduction. We will use the following notion and notations. We mainly refer to [3]; see also [1].

Let T be a closed linear operator in a Banach space E, $\rho(T)$ its resolvent set, and $\sigma(T)$ its spectrum. The dimension of the null space of T, $N(T)$, written $\alpha(T)$, will be called the *kernel index* of T and the *deficiency* of the range T in E, $R(T)$, written $\beta(T)$, will be called the *deficiency index* of T. The *index* $\kappa(T)$ is defined by

$$\kappa(T)=\alpha(T)-\beta(T).$$

If the operator T has a finite index, it is called a *Fredholm operator*.

We denote by $\sigma_{em}(T)$ the set of all complex number λ for which $T-\lambda I$ is not a Fredholm operator with index zero and call it the *essential spectrum* of T. The set of points of $\sigma(T)$ which is not an isolated eigenvalue λ of finite multiplicity, namely $\alpha(T-\lambda I)<\infty$, will be denoted by $\sigma_0(T)$. Here an isolated eigenvalue means an eigenvalue which is an isolated point of the spectrum.

Let X be a Banach space and H a Hilbert space such that

i) *$X\subset H$, and the embedding mapping; $X\to H$ is continuous,*

ii) *X is dense in H.*

The purpose of this paper is to prove the following theorem:

Theorem. *Let T be a closed linear operator in X and essentially self-adjoint in H, that is, its smallest closed extension* (*or its closure*) *in H is self-adjoint. Then*

$$\sigma_0(T\mid X)=\sigma_{em}(T\mid X).$$

Here we denoted by $T\mid X$ the operator considered in X. Similarly we will denote by $\bar{T}$ the closure of T in H, $\sigma(\bar{T}\mid H)$ the spectrum of $\bar{T}$ in H and so on.

Since the index of the Fredholm operator is invariant under the addition of compact operators [3, Theorem V.2.1], in particular, when K is a linear compact operator in X,

$$\sigma_{em}(T\mid X)=\sigma_{em}(T+K\mid X).$$

In addition, if K is symmetrizable, that is, symmetric with respect to the inner product of H, $T+K$ is essentially self-adjoint [4, p. 288, Theorem 4.4] and, by our Theorem

$$\sigma_0(T|X)=\sigma_0(T+K|X).$$

When $H=X$, this relation is equivalent to the classical H. Weyle's theorem.

J. Nieto proved the above Theorem in the case when T is bounded in X [6]. We extended it to the unbounded operators (see also latter Remark) and eliminate the needless part in his proof.

2. Basic Propositions and Lemmas. To prove Theorem we need some propositions and lemmas.

Proposition 1 [3, Theorem V.1.6 and 1.8]. *Let T be a closed operator on a Banach space E and $\Phi_T(E)$ be the Fredholm set; the set of all complex number λ for which $T-\lambda I$ is a Fredholm operator. Then Φ_T is open in the complex plane and the index $\kappa(T-I)$ is constant on the connected component of Φ_T. The kernel index $\alpha(T-\lambda I)$ is also constant on it, with the possible exception of isolated points.*

Proposition 2 [8, Theorem 9.2 and 9.6]. *Let λ_0 be an isolated point of $\sigma(T)$ and P_{λ_0} be the corresponding spectral projection operator;*

$$P_{\lambda_0}x=\frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\varepsilon}\frac{x}{\lambda I-T}d\lambda, \qquad x\in E$$

for sufficiently small positive ε. Then the dimension of the range of P_{λ_0} is finite if and only if $\kappa(T-\lambda_0 I)$ is zero.

Lemma 1 [5]. *Let H and X be the spaces as in section 1. If T is a bounded operator in X and symmetrizable, it is also bounded in H (with respect to the norm of H).*

Lemma 2. *If the closed operator T in X is essentially self-adjoint, then*

$$\sigma(\bar{T}|H)\subset\sigma(T|X).$$

Proof. Let λ_0 be a real number belonging to the resolvent of $T|X$. $T-\lambda_0 I$ is essentially self-adjoint and has a bounded right inverse S in X. By Lemma 1 S is also bounded in H, so its closure $\bar{S}$ in H is defined on all of H and is a right inverse of $\bar{T}-\lambda_0 I$ in H. This completes the proof of Lemma 2.

3. Proof of Theorem. We are now in position to prove Theorem.

a) We first prove that if λ_0 is an isolated eigenvalue of finite multiplicity, $\kappa(T-\lambda_0 I|X)$ is zero. The eigenvalue of $T|X$ is also the eigenvalue of $\bar{T}|H$, since $H\supset X$. Hence by the relation of Lemma 2, the isolated eigenvalue λ_0 of $T|X$ is the isolated eigenvalue of $\bar{T}|H$. Since $\bar{T}|H$ is self-adjoint the eigenvalue λ_0 is a simple pole of $\bar{T}$. If we denote by $P_{\lambda_0|H}$ the projection associated with $\{\lambda_0\}$ of $\bar{T}|H$,

$$(\bar{T}-\lambda_0 I)P_{\lambda_0|H}=0, \qquad \text{for all } x\in H.$$

The restriction of $P_{\lambda_0|H}$ to X, written $P_{\lambda_0|X}$, is the corresponding spectral projection of $T|X$ and

$$(T-\lambda_0 I)P_{\lambda_0|X}x=0, \qquad \text{for all } x\in X.$$

Hence the range of $P_{\lambda_0|X}$ is contained in $N(T-\lambda_0 I|X)$, the null space of $T-\lambda_0 I$ in X and by the assumption the dimension of $N(T-\lambda_0 I|X)$, namely $\alpha(T-\lambda_0 I|X)$ is finite; therefore $\beta(P_{\lambda_0|X})$ is finite. By Proposition 2, it follows that $\kappa(T-\lambda_0 I|X)$ is zero.

We note that λ_0 is shown to be a simple pole of $T|X$.

b) Now we prove the converse of a).

Let $\lambda_0\in\sigma(T|X)$ and $\kappa(T-\lambda_0 I|X)=0$. By Proposition 1, $\kappa(T-\lambda I|X)$ and $\alpha(T-\lambda I|X)$ are constant for any $\lambda\in V-\{\lambda_0\}$, where V is some neighbourhood of λ_0. Since there exists λ' of V belonging to the resolvent of $\bar{T}|H$,

$N(\bar{T}-\lambda' I|H)=\{0\}$, which shows that $N(T-\lambda' I|X)=\{0\}$, namely $\alpha(T-\lambda' I|X)=0$. Hence $\alpha(T-\lambda I|X)=0$, for any $\lambda\in V-\{\lambda_0\}$; consequently $\beta(T-\lambda I|X)=0$, by the assumption that $\kappa(T-\lambda I|X)=0$. Thus we obtain the relation that

$$V-\{\lambda_0\}\subset\rho(T|X).$$

This implies that λ_0 is an isolated eigenvalue of $T|X$.

4. Remark. Under the weaker assumptions, Theorem holds.

Lemma 1 holds when the operator T is faithful; for the adjoint operator T^* of $T|H$, T^*X is contained in X [2].

If any isolated spectrum of $T|X$ is the isolated spectrum of $\bar{T}|H$, and the isolated eigenvalues of $\bar{T}|H$ are all poles, the part a) of the proof of Theorem is similarly carried out. In fact an isolated eigenvalue of $T|X$ is also isolated for $\bar{T}|H$ since by the above remark Lemma 2 holds, and by the following Proposition the latter half of the proof also holds.

Proposition 3 [7, Theorem 5.8–A]. *If λ_0 is a pole of degree m of an operator T in a Banach space E,*

$$(T-\lambda_0 I)^m P_{\lambda_0}x=0, \qquad \text{for all } x \text{ of } E,$$

and dimension of $N((T-\lambda_0 I)^m)\leqq m\alpha(T-\lambda_0 I)$.

It is also easily seen that the part b) of the proof of Theorem holds if the resolvent set of $\bar{T}|H$ is dense in the complex plane.

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