3. *On the Representations of Semi-Simple Lie Groups*

By Shôichirô SAKAI
Mathematical Institute, Tôhoku University, Sendai
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The purpose of this note is to give some remarks on the representations of semi-simple Lie groups. In this note we give only the Definitions and Theorems, since we shall give discussions elsewhere in detail.

Let G be a connected Lie group, and $C_c^\infty(G)$ be the algebra composed of indefinitely differentiable complex-valued functions with compact supports.

Let $U(G)$ be the subalgebra of $D(G)$ composed of all elements whose supports reduce to the identity, then $U(G)$ is isomorphic to the universal enveloping algebra B[1] corresponding to G.

Let $D_z(G)$ be the center of $D(G)$ and $\varepsilon_s(s \in G)$ be the point measure with mass 1 at s.

Then we can easily show that $\alpha(\in D(G))$ belongs to $D_z(G)$ if and only if $\varepsilon_s \alpha \varepsilon_{s^{-1}} = \alpha$ for all $s \in G$. Let $U_z(G)$ be the center of $U(G)$, then $U_z(G) \subset D_z(G)$.

Let $\{\Pi(x), \mathfrak{H}\}$ be a strongly continuous representation of G on a Banach space $\mathfrak{H}$ and $\{\Pi(f), \mathfrak{H}\}$ be the corresponding representation of $C_c^\infty(G)$. Let $\boldsymbol{B}$ be the operator algebra composed of all bounded operators on $\mathfrak{H}$. We shall state

Definition 1. A representation $\{\Pi(x), \mathfrak{B}\}$ is n-fold irreducible, if there exists an element $\Pi(f)$ such that

$$\|\Pi(f)x_i - Bx_i\| < \varepsilon \qquad (i = 1, 2, \dots, n)$$

for arbitrary at most n elements $x_1, \dots, x_n$, $B \in \boldsymbol{B}$ and $\varepsilon > 0$.

Proposition. If $\{\Pi(x), \mathfrak{H}\}$ is 2-fold irreducible, it is quasi-simple.[2]

In the following, we shall suppose that G is a connected semi-simple Lie group with a decomposition $G = K \cdot S (K \cap S = (e))$ where K is a maximal compact subgroup and S is a quasi-nilpotent subgroup[3] in the sense of Harish-Chandra.[3] Since the above condition i.e. $G = K \cdot S$, seems to be indispensable at certain essential points in our note, we have decided for the sake of uniformity to assume it throughout. Let P be the set of all equivalence classes of irreducible representations of K and $\chi_d(k)$ be the character of $d(\in P)$.

We shall denote the equivalence class of irreducible representation of $U(K)$ which corresponds to $d(\in P)$ by the same notation d.

Lemma 1. Let $\varphi(x)$ be an analytic function on G and μ be a Radon measure with a compact support. Then $(\mu\varphi)(x)$ and $(\varphi\mu)(x)$ are analytic functions.

Lemma 2. Let $\{U(x), L^2(G)\}$ be the left regular representation of G. If $\alpha\in D(G)$ satisfies

$$\int < U(x)e,\ f > d\alpha(x) = 0$$

($< >$ is the scalar product of $L^2(G)$) for all analytic coefficients, then $\alpha=0$. (Cf. (3) Theorem 3. p. 20.)

Let A be a vector space composed of all analytic function on G.

Definition 2. We define that a variable $\alpha\in D(G)$ converges to $\alpha_0\in D(G)$ if $\varphi(\alpha)$[4] converges to $\varphi(\alpha_0)$ for all $\varphi\in A$.

Then, by Lemma 2, $D(G)$ is a locally convex topological vector space and $U(G)$ is everywhere dense in $D(G)$.

We shall consider that $U(G)$ and $C_c^\infty(G)$ are locally convex topological vector spaces by the relative topologies.

Let $\{\Pi(x), \mathfrak{H}\}$ be a strongly continuous representation of G on a Banach space $\mathfrak{H}$ and $\{\Pi(f), \mathfrak{H}\}$ be the corresponding representation of $C_c^\infty(G)$, V be the Gårding subspace[13] of $\mathfrak{H}$. Then we obtain a representation $\{\Pi, V\}$ of $D(G)$ on V.[9]

We shall put

Definition 3. A representation is strongly cyclic if it satisfies

$$[\Pi(G)e]^{5)} \quad \text{for some} \quad e\in\sum_{d\in p} V(d).^{6)}$$

Moreover, in this case e is called a strongly cyclic vector.

Let $\{\Pi(x), \mathfrak{H}\}$ be a strongly cyclic representation with an infinitesimal character.[2] Then by Harish-Chandra's theorem $[\Pi(U(G))e]=\mathfrak{H}$ and $\Pi(U(G))e=\sum_{d\in P}\mathfrak{H}(d)$, $\dim\mathfrak{H}(d)<\infty$ for all $d\in P$. Put $\mathfrak{M}_e=\{\alpha\,|\,\Pi(\alpha)e=0\ \ \alpha\in U(G)\}$, then $\mathfrak{M}_e$ is a closed left ideal in $U(G)$, and $(U(G)/\mathfrak{M}_e)=\sum_{d\in P}(U(G)/\mathfrak{M})(d)$ and $\dim(U(G)/\mathfrak{M}_e)(d)<\infty$.

Definition 4. We say that a closed left ideal $\mathfrak{M}$ of $U(G)$ is an F-left ideal if it satisfies

$$(U(G)/\mathfrak{M})=\sum_{d\in P}(U(G)/\mathfrak{M})(d) \quad\text{and}\quad \dim(U(G)/\mathfrak{M})(d)<\infty\ .$$

Lemma 3. Define linear operator L_α, $R_\alpha(\alpha\in U(G)$ or $M^{7)}(G))$ on $D(G)$ as follows: $L_\alpha\beta=\alpha\beta$ and $R_\alpha\beta=\beta\alpha$. Then L_α and R_α are continuous.

Theorem 1. Let $\mathfrak{M}$ be a closed left ideal of $U(G)$, then $\mathfrak{M}^{b}$[8] is an invariant subspace of $D(G)$ under $M(G)$ and $U(G)$, and $\mathfrak{M}^b\frown C_c^\infty(G)$ is a closed left-ideal and is invariant under $M(G)$ and $U(G)$, and $(\mathfrak{M}^b\frown C_c^\infty(G))^b\frown U(G)=\mathfrak{M}$. Moreover if $\mathfrak{M}$ is an F-left ideal and $\mathfrak{N}$ is a closed left ideal of $C_c^\infty(G)$ and $\mathfrak{N}^b\frown U(G)=\mathfrak{M}$, then $(\mathfrak{N}^b\frown U(G))^b\frown C_c^\infty(G)=\mathfrak{N}$ and $\mathfrak{N}$ is a regular left ideal, and furthermore in a representation $\{\Pi_{\mathfrak{N}}, C_c^\infty(G)/\mathfrak{N}\}$ of G on a topological vector space $C_c^\infty(G)/\mathfrak{N}$, the corresponding representation, $\{\Pi_{\mathfrak{N}}, W_1 = \sum_{d\in P}(C_c^\infty(G)/\mathfrak{N})(d)\}$ of $U(G)$ is equivalent to the canonical representation $\{\Pi_{\mathfrak{M}}, U(G)/\mathfrak{M}\}$. In particular if $\mathfrak{M}$ is a maximal F-left ideal, $\mathfrak{N}$ is maximal.

Here we shall sketch the proof of Theorem 1. From the density of $U(G)$ and the continuity of L_α, $R_\alpha(\alpha\in M(G)$ or $U(G))$, we can easily see that $(C_c^\infty(G)\mathfrak{M})^b = (M(G)\mathfrak{M})^b = (D(G)\mathfrak{M})^b = \mathfrak{M}^b$, and so $\mathfrak{M}^b$ and $\mathfrak{M}^b\frown C_c^\infty(G)$ are invariant under $M(G)$ and $U(G)$, and moreover $(\mathfrak{M}^b\frown C_c^\infty(G))^b\frown U(G) = \mathfrak{M}$.

In particular let $\mathfrak{M}$ be an F-left ideal and $\alpha\to\alpha_\rho$ be the natural mapping from $U(G)$ on $U(G)/\mathfrak{M}$.

Then if $\alpha\rho\in U(G)/\mathfrak{M}(d')$, $\overline{\chi}_d\alpha$ transforms according to d' in the space $D(G)/\mathfrak{M}^b$. On the other hand it is clear that $\overline{\chi}_d\alpha$ transforms according to d in the space $D(G)/\mathfrak{M}^b$.

Hence if $d\neq d'$, $\overline{\chi}_d\alpha\equiv 0 \pmod{\mathfrak{M}^b}$ and so $(\overline{\chi}_d U(G)+\mathfrak{M}^b)/\mathfrak{M}^b$, is finite-dimensional.

From some additional considerations with the above facts, we can show Theorem 1.

Remark. To imbed the above representation $\{\Pi_{\mathfrak{N}}, C_c^\infty(G)/\mathfrak{N}\}$ of G into a representation on a Banach space seems to be interesting. However the author could not show this fact without some additional conditions.

Corollary 1. Let $\{\Pi(x), \mathfrak{H}\}$ be a strongly cyclic representation with an infinitesimal character on a Banach space $\mathfrak{H}$, and $\mathfrak{M} = \{\alpha\,|\,\Pi(\alpha)e = 0,\ \alpha\in U(G)\}$ and $\mathfrak{N} = \{f\,|\,\Pi(f)e = 0,\ f\in C_c^\infty(G)\}$, where e is a strongly cyclic vector. Then $\mathfrak{N}$ is a regular closed left ideal, and $\mathfrak{M}^b\frown C_c^\infty(G) = \mathfrak{N}$ and $\mathfrak{N}^b\frown U(G) = \mathfrak{M}$. Moreover if $\{\Pi(x), \mathfrak{H}\}$ is irreducible, $\mathfrak{N}$ is maximal. (Cf. (2) and Godement,[9] Theorem 6, p. 513.) We shall state

Definition 5. A continuous linear functional on $U(G)$ is a state if it satisfies

$$\varphi(\alpha^{*10)}\alpha) \geqq 0 \quad \text{for all} \quad \alpha\in U(G).$$

It is clear that if $\psi(x)$ is an analytic positive definite function i.e. $\psi(x)$ is analytic and $\psi(\gamma^*\gamma)\geqq 0$ for all $\gamma\in M(G)$ then it can be considered to be a state.

Let φ be a state and $\mathfrak{M}_\varphi = \{\alpha\,|\,\varphi(\alpha^*\alpha) = 0,\ \alpha\in U(G)\}$, then $\mathfrak{M}_\varphi$ is a closed left ideal of $U(G)$. We shall say $\mathfrak{M}_\varphi$ to be the kernel of φ.

Definition 6. We say that a state is an F-state if it has an F-left ideal as the kernel.

Theorem 2. If φ is an F-state on $U(G)$, then it is an analytic positive definite function on G, and if $\mathfrak{M}_\varphi$ is the kernel of φ, the canonical representation $\{\Pi_{\mathfrak{M}_\varphi}, U(G)/\mathfrak{M}_\varphi\}$ of $U(G)$ is equivalent to the representation of $U(G)$ corresponding to a unitary representation $\{\Pi_\varphi, \mathfrak{H}_\varphi\}$[11] constructed by φ.

Corollary 2. In order that a spherical function[2)9)] $\varphi_d^\Pi(x)$ is positive definite, it is necessary and sufficient that it satisfies $\varphi_d^\Pi(\alpha^*\alpha) \geqq 0$ for all $\alpha \in U(G)$.

Corollary 3. If $\mathfrak{M}$ is an F-left ideal and $\mathfrak{M}'(\supset \mathfrak{M})$ is a left ideal, then $\mathfrak{M}'$ is an F-left ideal.

Lemma 4. Let $\{\Pi(x), \mathfrak{H}\}$ be an irreducible representation with an infinitesimal character of G on a Banach space. Put $W = \sum_{d \in P} \mathfrak{H}(d)$, then for an arbitrary linear transformation T in W with a finite-dimensional domain $\mathfrak{D}(T)$, there exists an operator $\Pi(f)$ $(f \in C_c^\infty(G))$ such that $\Pi(f) = T$ on $\mathfrak{D}(T)$.

From Lemma 4, we obtain the following theorem.

Theorem 3. In order that an irreducible representation $\{\Pi(x), \mathfrak{H}\}$ with an infinitesimal character is infinitesimally equivalent to a unitary irreducible representation on a Hilbert space, it is necessary and sufficient that a spherical function $\varphi_d^\Pi(x) (\neq 0)$ is positive definite.

From Corollary 2 and Theorem 3 we can easily show the following result of Harish-Chandra :[2]

Theorem (Harish-Chandra[2]). Let $\{\Pi(x), \mathfrak{H}\}$ be an irreducible representation with an infinitesimal character of G on a Banach space. Put $W = \sum_{d \in P} \mathfrak{H}(d)$. Suppose it is possible to define a new scalar product $(\ ,\ )'$ in W such that

$$(\Pi(\alpha)\varphi, \psi)' = (\varphi, \Pi(\alpha^*)\psi)' (\varphi, \psi \in W, \alpha \in \mathfrak{G}_0).$$ [12]

Let $\mathfrak{H}'$ be the Hilbert space obtained by completing W with the corresponding metric. Then there exists an irreducible unitary representation Π' of G on $\mathfrak{H}'$ such that

$$\Pi(\alpha)\psi = \lim_{t \to 0} \frac{1}{t} \{\Pi'(\exp t\alpha)\psi - \psi\}, \ (t \in R, \text{lim. in } \mathfrak{H}')$$

for all $\alpha \in \mathfrak{G}_0$, and $\psi \in W$. Moreover Π' is uniquely determined. For, from the property of the scalar product $(\ ,\ )'$, we can immediately show that $\varphi_d^\Pi(\alpha^*\alpha) = S_p(E(d)\Pi(\alpha^*\alpha)E(d)) \geqq 0$ for all $\alpha \in U(G)$.

Remark. We can easily show that the above Theorem of Harish-Chandra can be altered by the following form which seems to be convenient than the above :

Let $\{\Pi(x), \mathfrak{H}\}$ be an irreducible representation of G with an infinitesimal character. Put $W=\sum_{d\in P}\mathfrak{H}(d)$. Suppose it is possible to define a new scalar product $(\ ,\ )'$ in W such that

(i) $(\Pi(\alpha)\varphi, \psi)'=(\varphi, \Pi(\alpha^*)\psi)'$ $(\varphi, \psi\in W,\ \alpha\in S_0$[14]$)$

(ii) $(\Pi(\beta)\varphi, \psi)'=(\varphi, \Pi(\beta^*)\psi)'$ $(\varphi, \psi\in\mathfrak{H}(d_0)\ (\neq(0))$

for some $d_0\in P$, $\beta\in K_0$.[15]

(iii) $\mathfrak{H}(d_0)$ is orthogonal to every $\mathfrak{H}(d)\ (d\neq d_0)$

with respect to the new scalar product.

Then $\{\Pi(x), \mathfrak{H}\}$ is infinitesimally equivalent to an irreducible unitary representation.

References

1) Harish-Chandra: On some applications of universal enveloping algebra of a semi-simple Lie algebra. Trans. Amer. Math. Soc., **70**, 2 (1952).

2) Harish-Chandra: On representations of semi-simple Lie groups. Proc. Nat. Acad. Sci. U.S.A., **37**, 170–173, 362–365, 366–369 (1951).

3) Harish-Chandra: Representations of a semi-simple Lie group on a Banach space, I. Trans. Amer. Math. Soc., **75**, 2 (1953).

4) $\varphi(\alpha)$ means $\int\varphi(x)d\alpha(x)$.

5) $[\pi(G)e]$ denotes the closed linear subspace generated by $\pi(G)e$.

6) $V(d)$ is a subspace of V composed of all elements which transform under $\pi(K)$ according to d.

7) $M(G)$ denotes the subalgebra of $D(G)$ composed of all Radon measures with compact supports.

8) "b" denotes the closure operation in $D(G)$.

9) R. Godement: A theory of spherical function, I. Trans. Amer. Math. Soc., **73**, 3 (1952).

10) α^* is defined as follows: $\alpha^*(f)=\alpha(f^*)$ for all $f\in C_c^\infty(G)$.

11) R. Godement: Les fonctions de type positif et la theorie des groupes. Trans. Amer. Math. Soc., **63** (1948).

12) $\mathfrak{G}_0$ denotes the Lie ring of G.

13) L. Gårding: Proc. Nat. Acad. Sci. U. S. A., **33**, 331–332 (1947).

14) S_0 denotes the Lie ring of the subgroup S.

15) K_0 denotes the Lie ring of the subgroup K.