135. Probabilities on Inheritance in Consanguineous Families. IX

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VIII. Combinations through the most extreme consanguineous marriages

1. Parents-descendants combinations

In the present and subsequence chapters, we shall supplement the results on combinations which have been postponed in VI, §1 as the extreme ones.

We first attempt to determine *the probability of parents-descendants combinations immediate after successive consanguineous marriages of the extreme mode, which will be designated by*

$$\mathfrak{e}_t(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2) \equiv \varepsilon_{(11;0)_{t-1}|11}(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2);$$

as to the notation cf. VI, §1.

It is readily seen that the quantity in consideration satisfies a recurrence equation

$$\mathfrak{e}_t(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2) = \sum \varepsilon(\alpha\beta, \gamma\delta; ab, cd)\mathfrak{e}_{t-1}(ab, cd; \xi_1\eta_1, \xi_2\eta_2)$$

with

$$\mathfrak{e}_1(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2) \equiv \varepsilon(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2),$$

where the summation extends, as usual, over all the possible pairs of (ab, cd). Though the quantity $\mathfrak{e}_t$ has originally been defined for $t \geqq 1$, it is convenient to define $\mathfrak{e}_0$ as follows: When $(\alpha\beta, \gamma\delta)$ coincides with none of $(\xi_1\eta_1, \xi_2\eta_2)$ and $(\xi_2\eta_2, \xi_1\eta_1)$, then $\mathfrak{e}_0(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2) = 0$; when $(\alpha\beta, \gamma\delta)$ coincides with $(\xi_1\eta_1, \xi_2\eta_2)$ or $(\xi_2\eta_2, \xi_1\eta_1)$, then $\mathfrak{e}_0 = 1$ or $\mathfrak{e}_0 = 1/2$ according to $A_{\alpha\beta} = A_{\gamma\delta}$ or $A_{\alpha\beta} \neq A_{\gamma\delta}$.

To determine the values of the $\mathfrak{e}_t$'s, we distinguish four systems according to the number of different genes contained in parents' types, based on a reason that possible types of descendants are restricted to those consisting of the genes involved in their parents' types. In fact, intermediate marriages under consideration are so extreme that there concern no individuals from other lineages.

In each system, the recurrence equation can be regarded, for a fixed pair of descendants' types as *a system of difference equations* of the first order in which the unknowns are the probabilities for possible pairs of parents' types. For the sake of brevity, we shall use for a while an abbreviation

$$[\alpha\beta, \gamma\delta]_t \equiv \mathfrak{e}_t(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2),$$

for a fixed pair $(\xi_1\eta_1, \xi_2\eta_2)$, unless any confusion can arise.

I. *Case of a single gene A_i.* There exists only one possible combination for which we readily get

$$\mathfrak{e}_t(ii, ii; ii, ii) = 1.$$

II. *Case of two genes A_i and A_k.* The possible pairs for $(\alpha\beta, \gamma\delta)$ amount, by virtue of an evident symmetry, essentially to three, i. e. (ii, ik), (ii, kk) and (ik, ik). Thus we get a system of difference equations with constant coefficients in the form

$$\begin{aligned}
[ii, ik]_t &= \tfrac{1}{4}[ii, ii]_{t-1} + \tfrac{1}{2}[ii, ik]_{t-1} + \tfrac{1}{4}[ik, ik]_{t-1}, \\
[ii, kk]_t &= [ik, ik]_{t-1}, \\
[ik, ik]_t &= \tfrac{1}{16}[ii, ii]_{t-1} + \tfrac{1}{4}[ii, ik]_{t-1} + \tfrac{1}{8}[ii, kk]_{t-1} + \tfrac{1}{4}[ik, ik]_{t-1} \\
&\quad + \tfrac{1}{4}[ik, kk]_{t-1} + \tfrac{1}{16}[kk, kk]_{t-1}.
\end{aligned}$$

III. *Case of three genes A_i, A_h and A_k.* A system of difference equations is then written in the form

$$\begin{aligned}
[ii, hk]_t &= \tfrac{1}{4}[ih, ih]_{t-1} + \tfrac{1}{2}[ih, ik]_{t-1} + \tfrac{1}{4}[ik, ik]_{t-1}, \\
[ih, ik]_t &= \tfrac{1}{16}[ii, ii]_{t-1} + \tfrac{1}{8}[ii, ih]_{t-1} + \tfrac{1}{8}[ii, ik]_{t-1} + \tfrac{1}{8}[ii, hk]_{t-1} \\
&\quad + \tfrac{1}{16}[ih, ih]_{t-1} + \tfrac{1}{8}[ih, ik]_{t-1} + \tfrac{1}{16}[ik, ik]_{t-1} + \tfrac{1}{8}[ih, hk]_{t-1} \\
&\quad + \tfrac{1}{8}[ik, hk]_{t-1} + \tfrac{1}{16}[hk, hk]_{t-1}, \\
[ik, hk]_t &= \tfrac{1}{16}[ih, ih]_{t-1} + \tfrac{1}{8}[ih, ik]_{t-1} + \tfrac{1}{16}[ik, ik]_{t-1} + \tfrac{1}{8}[ih, hk]_{t-1} \\
&\quad + \tfrac{1}{8}[ih, kk]_{t-1} + \tfrac{1}{8}[ik, hk]_{t-1} + \tfrac{1}{8}[ik, kk]_{t-1} + \tfrac{1}{16}[hk, hk]_{t-1} \\
&\quad + \tfrac{1}{8}[hk, kk]_{t-1} + \tfrac{1}{16}[kk, kk]_{t-1}.
\end{aligned}$$

Here the equations obtainable by merely interchanging the letters are omitted.

IV. *Case of four genes A_i, A_j, A_h and A_k.* A system of difference equations is then expressed, by virtue of symmetry, by a single representative

$$\begin{aligned}
[ij, hk]_t &= \tfrac{1}{16}[ih, ih]_{t-1} + \tfrac{1}{8}[ih, jh]_{t-1} + \tfrac{1}{16}[jh, jh]_{t-1} + \tfrac{1}{16}[ik, ik]_{t-1} \\
&\quad + \tfrac{1}{8}[ik, jk]_{t-1} + \tfrac{1}{16}[jk, jk]_{t-1} + \tfrac{1}{8}[ih, ik]_{t-1} + \tfrac{1}{8}[jh, jk]_{t-1} \\
&\quad + \tfrac{1}{8}[ih, jk]_{t-1} + \tfrac{1}{8}[jh, ik]_{t-1}.
\end{aligned}$$

In each system, the equations should be solved for every pair of descendants' types with suitable initial conditions. The characteristic roots of indicial equations for these systems of difference equations are contained in the set 1, 1/2, 1/4, $-1/4$, $-1/8$, ω and $\tilde{\omega}$, where *ω and $\tilde{\omega}$ denote conjugate irrational numbers satisfying a quadratic equation* $4\zeta^2 - 2\zeta - 1 = 0$:

$$\omega = \frac{1+\sqrt{5}}{4}, \qquad \tilde{\omega} = \frac{1-\sqrt{5}}{4}.$$

We now introduce a symbol $\boldsymbol{R}$ which designates the "*rational part*" of an irrational number in a field obtained from the field of rational numbers by adjoining $\sqrt{5}$. For instance, we put

$$\boldsymbol{R}B\omega^t = \tfrac{1}{2}(B\omega^t + \tilde{B}\tilde{\omega}^t),$$

$\tilde{B}$ being the number in the field conjugate to B.

In the following lines, we set out the values of the $\mathfrak{e}_t$'s which cover essentially all the possible combinations:

$$e_t(ii,\ ii;\ ii,\ ii)=1;$$

$$e_t(ii,\ ik;\ ii,\ ii)=\frac{3}{4}-\boldsymbol{R}\frac{9+4\sqrt{5}}{20}\omega^t-\frac{1}{4}\frac{1}{2^t}-\frac{1}{20}\frac{1}{4^t},$$

$$e_t(ii,\ ik;\ ii,\ ik)=\boldsymbol{R}\frac{3+\sqrt{5}}{20}\omega^t+\frac{1}{4}\frac{1}{2^t}+\frac{1}{10}\frac{1}{4^t},$$

$$e_t(ii,\ ik;\ ii,\ kk)=\boldsymbol{R}\frac{1}{20}\omega^t-\frac{1}{20}\frac{1}{4^t},$$

$$e_t(ii,\ ik;\ ik,\ ik)=\boldsymbol{R}\frac{1+\sqrt{5}}{5}\omega^t-\frac{1}{5}\frac{1}{4^t},$$

$$e_t(ii,\ ik;\ ik,\ kk)=\boldsymbol{R}\frac{3+\sqrt{5}}{20}\omega^t-\frac{1}{4}\frac{1}{2^t}+\frac{1}{10}\frac{1}{4^t},$$

$$e_t(ii,\ ik;\ kk,\ kk)=\frac{1}{4}-\boldsymbol{R}\frac{9+4\sqrt{5}}{20}\omega^t+\frac{1}{4}\frac{1}{2^t}-\frac{1}{20}\frac{1}{4^t};$$

$$e_t(ii,\ kk;\ ii,\ ii)=\frac{1}{2}-\boldsymbol{R}\frac{7+3\sqrt{5}}{10}\omega^t+\frac{1}{5}\frac{1}{4^t},$$

$$e_t(ii,\ kk;\ ii,\ ik)=\boldsymbol{R}\frac{2}{5}\omega^t-\frac{2}{5}\frac{1}{4^t},$$

$$e_t(ii,\ kk;\ ii,\ kk)=\boldsymbol{R}\frac{3-\sqrt{5}}{10}\omega^t+\frac{1}{5}\frac{1}{4^t},$$

$$e_t(ii,\ kk;\ ik,\ ik)=\boldsymbol{R}\frac{-4+4\sqrt{5}}{5}\omega^t+\frac{4}{5}\frac{1}{4^t};$$

$$e_t(ik,\ ik;\ ii,\ ii)=\frac{1}{2}-\boldsymbol{R}\frac{11+5\sqrt{5}}{20}\omega^t+\frac{1}{20}\frac{1}{4^t},$$

$$e_t(ik,\ ik;\ ii,\ ik)=\boldsymbol{R}\frac{1+\sqrt{5}}{10}\omega^t-\frac{1}{10}\frac{1}{4^t},$$

$$e_t(ik,\ ik;\ ii,\ kk)=\boldsymbol{R}\frac{-1+\sqrt{5}}{20}\omega^t+\frac{1}{20}\frac{1}{4^t},$$

$$e_t(ik,\ ik;\ ik,\ ik)=\boldsymbol{R}\frac{4}{5}\omega^t+\frac{1}{5}\frac{1}{4^t};$$

$$e_t(ii,\ hk;\ ii,\ ik)=\boldsymbol{R}\frac{1}{5}\omega^t-\frac{1}{5}\frac{1}{4^t},$$

$$e_t(ii,\ hk;\ ii,\ kk)=\boldsymbol{R}\frac{3-\sqrt{5}}{20}\omega^t-\frac{1}{60}\frac{1}{2^t}+\frac{1}{60}\frac{1}{4^t}-\frac{1}{12}\frac{1}{(-4)^t}-\frac{1}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ ii,\ hk)=\frac{1}{30}\frac{1}{2^t}+\frac{1}{6}\frac{1}{4^t}+\frac{1}{6}\frac{1}{(-4)^t}+\frac{2}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ ik,\ ik)=\boldsymbol{R}\frac{-2+2\sqrt{5}}{5}\omega^t-\frac{2}{15}\frac{1}{2^t}+\frac{1}{15}\frac{1}{4^t}+\frac{1}{3}\frac{1}{(-4)^t}+\frac{2}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ ih,\ ik)=\frac{2}{15}\frac{1}{2^t}+\frac{1}{3}\frac{1}{4^t}-\frac{1}{3}\frac{1}{(-4)^t}-\frac{2}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ ik,\ kk)=\boldsymbol{R}\frac{1}{5}\omega^t-\frac{1}{6}\frac{1}{2^t}+\frac{1}{20}\frac{1}{4^t}-\frac{1}{12}\frac{1}{(-4)^t},$$

$$e_t(ii,\ hk;\ ik,\ hh)=\frac{1}{30}\frac{1}{2^t}-\frac{1}{12}\frac{1}{4^t}-\frac{1}{12}\frac{1}{(-4)^t}+\frac{2}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ ik,\ hk)=\frac{2}{15}\frac{1}{2^t}-\frac{1}{6}\frac{1}{4^t}+\frac{1}{6}\frac{1}{(-4)^t}-\frac{2}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ hh,\ kk)=\boldsymbol{R}\frac{-2+\sqrt{5}}{20}\omega^t-\frac{1}{60}\frac{1}{2^t}+\frac{1}{60}\frac{1}{4^t}+\frac{1}{6}\frac{1}{(-4)^t}-\frac{1}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ hk,\ hk)=\boldsymbol{R}\frac{3-\sqrt{5}}{5}\omega^t-\frac{2}{15}\frac{1}{2^t}+\frac{1}{15}\frac{1}{4^t}-\frac{2}{3}\frac{1}{(-4)^t}+\frac{2}{15}\frac{1}{(-8)^t},$$

$$e_t(ii,\ hk;\ hk,\ kk)=\boldsymbol{R}\frac{-1+\sqrt{5}}{20}\omega^t-\frac{1}{12}\frac{1}{2^t}+\frac{1}{20}\frac{1}{4^t}+\frac{1}{12}\frac{1}{(-4)^t};$$

$$e_t(ij,\ ik;\ ii,\ ij)=\boldsymbol{R}\frac{1+\sqrt{5}}{20}\omega^t-\frac{1}{20}\frac{1}{4^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ii,\ jj)= \boldsymbol{R}\frac{-1+\sqrt{5}}{40}\omega^t -\frac{1}{60}\frac{1}{2^t}-\frac{1}{60}\frac{1}{4^t}+\frac{1}{24}\frac{1}{(-4)^t}+\frac{1}{60}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ii,\ jk)= \frac{1}{30}\frac{1}{2^t}+\frac{1}{12}\frac{1}{4^t}-\frac{1}{12}\frac{1}{(-4)^t}-\frac{1}{30}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ij,\ ij)= \boldsymbol{R}\frac{2}{5}\omega^t -\frac{2}{15}\frac{1}{2^t}-\frac{1}{15}\frac{1}{4^t}-\frac{1}{6}\frac{1}{(-4)^t}-\frac{1}{30}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ij,\ jj)= \boldsymbol{R}\frac{1+\sqrt{5}}{20}\omega^t -\frac{1}{6}\frac{1}{2^t}+\frac{3}{40}\frac{1}{4^t}+\frac{1}{24}\frac{1}{(-4)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ij,\ ik)= \frac{2}{15}\frac{1}{2^t}+\frac{1}{6}\frac{1}{4^t}+\frac{1}{6}\frac{1}{(-4)^t}+\frac{1}{30}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ij,\ jk)= \frac{2}{15}\frac{1}{2^t}-\frac{1}{12}\frac{1}{4^t}-\frac{1}{12}\frac{1}{(-4)^t}+\frac{1}{30}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ij,\ kk)= \frac{1}{30}\frac{1}{2^t}-\frac{1}{24}\frac{1}{4^t}+\frac{1}{24}\frac{1}{(-4)^t}-\frac{1}{30}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ jj,\ jk)= \boldsymbol{R}\frac{1}{10}\omega^t -\frac{1}{12}\frac{1}{2^t}+\frac{1}{40}\frac{1}{4^t}-\frac{1}{24}\frac{1}{(-4)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ jj,\ kk)= \boldsymbol{R}\frac{3-\sqrt{5}}{40}\omega^t -\frac{1}{60}\frac{1}{2^t}+\frac{1}{120}\frac{1}{4^t}-\frac{1}{12}\frac{1}{(-4)^t}+\frac{1}{60}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ ik,\ jk)= \frac{2}{15}\frac{1}{2^t}-\frac{1}{12}\frac{1}{4^t}-\frac{1}{12}\frac{1}{(-4)^t}+\frac{1}{30}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ ik;\ jk,\ jk)= \boldsymbol{R}\frac{-1+\sqrt{5}}{5}\omega^t -\frac{2}{15}\frac{1}{2^t}+\frac{1}{30}\frac{1}{4^t}+\frac{1}{3}\frac{1}{(-4)^t}-\frac{1}{30}\frac{1}{(-8)^t};$$

$$\mathfrak{e}_t(ij,\ hk;\ ii,\ jk)= \frac{1}{60}\frac{1}{2^t}-\frac{1}{24}\frac{1}{4^t}-\frac{1}{24}\frac{1}{(-4)^t}+\frac{1}{15}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ hk;\ ii,\ hk)= \frac{1}{60}\frac{1}{2^t}+\frac{1}{12}\frac{1}{(-4)^t}-\frac{1}{10}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ hk;\ ij,\ ik)= \frac{1}{15}\frac{1}{2^t}-\frac{1}{12}\frac{1}{4^t}+\frac{1}{12}\frac{1}{(-4)^t}-\frac{1}{15}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ hk;\ ij,\ hk)= \frac{1}{6}\frac{1}{4^t}+\frac{1}{3}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ hk;\ ik,\ jk)= \frac{1}{15}\frac{1}{2^t}-\frac{1}{6}\frac{1}{(-4)^t}+\frac{1}{10}\frac{1}{(-8)^t},$$

$$\mathfrak{e}_t(ij,\ hk;\ ik,\ jh)= -\frac{1}{6}\frac{1}{(-8)^t}.$$

2. Parents-descendant combinations

The probability of parents-descendants combinations having been explicitly established in the preceding section, we now deal with the probability of corresponding *parents-descendant combinations*. We first consider a combination *immediate after successive consanguineous marriages*. Its probability is given by an equation

$$\mathfrak{e}_{t-1|1}(\alpha\beta,\ \gamma\delta;\ \xi\eta)\equiv\varepsilon_{\langle 11;0\rangle_{t-1}|1}(\alpha\beta,\ \gamma\delta;\ \xi\eta)=\sum\mathfrak{e}_{t-1}(\alpha\beta,\ \gamma\delta;\ ab,\ cd)\varepsilon(ab,\ cd;\ \xi\eta)$$

or alternatively by

$$\mathfrak{e}_{t-1|1}(\alpha\beta,\ \gamma\delta;\ \xi\eta)=\sum\mathfrak{e}_t(\alpha\beta,\ \gamma\delta;\ \xi\eta,\ ab).$$

By making use of the values determined above for the summand, we get the results which will be enumerated in the following lines:

$$\mathfrak{e}_{t-1|1}(ii,\ ii;\ ii)=1;$$

$$\mathfrak{e}_{t-1|1}(ii,\ ik;\ ii)=\frac{3}{4}-\boldsymbol{R}\frac{5+3\sqrt{5}}{20}\omega^t,\qquad \mathfrak{e}_{t-1|1}(ii,\ ik;\ ik)=\boldsymbol{R}\frac{5+3\sqrt{5}}{10}\omega^t,$$

$$\mathfrak{e}_{t-1|1}(ii,\ ik;\ kk)=\frac{1}{4}-\boldsymbol{R}\frac{5+3\sqrt{5}}{20}\omega^t;$$

$$\mathfrak{e}_{t-1|1}(ii,\, kk;\, ii)=\frac{1}{2}-\boldsymbol{R}\frac{2\sqrt{5}}{5}\omega^t, \qquad \mathfrak{e}_{t-1|1}(ii,\, kk;\, ik)=\boldsymbol{R}\frac{4\sqrt{5}}{5}\omega^t;$$

$$\mathfrak{e}_{t-1|1}(ii,\, hk;\, ik)=\boldsymbol{R}\frac{2\sqrt{5}}{5}\omega^t, \qquad \mathfrak{e}_{t-1|1}(ii,\, hk;\, hk)=\boldsymbol{R}\frac{5-\sqrt{5}}{10}\omega^t;$$

$$\mathfrak{e}_{t-1|1}(ij,\, ij;\, ii)=\frac{1}{2}-\boldsymbol{R}\frac{5+\sqrt{5}}{10}\omega^t, \qquad \mathfrak{e}_{t-1|1}(ij,\, ij;\, ij)=\boldsymbol{R}\frac{5+\sqrt{5}}{5}\omega^t;$$

$$\mathfrak{e}_{t-1|1}(ij,\, ik;\, ij)=\boldsymbol{R}\frac{5+\sqrt{5}}{10}\omega^t, \qquad \mathfrak{e}_{t-1|1}(ij,\, ik;\, jk)=\boldsymbol{R}\frac{\sqrt{5}}{5}\omega^t.$$

The probability of a combination *distant after the last marriage* may be defined, for any $n>1$, by an equation

$$\mathfrak{e}_{t-1|n}(\alpha\beta,\, \gamma\delta;\, \xi\eta)\equiv\varepsilon_{(11;0)_{t-1}|n}(\alpha\beta,\, \gamma\delta;\, \xi\eta)=\sum \mathfrak{e}_{t-1}(\alpha\beta,\, \gamma\delta;\, ab)\kappa_{n-1}(ab;\, \xi\eta)$$

which leads to a simple result

$$\mathfrak{e}_{t-1|n}(\alpha\beta,\, \gamma\delta;\, \xi\eta)=\varepsilon_n(\alpha\beta,\, \gamma\delta;\, \xi\eta);$$

i. e. a similar rule as stated in V, §4 remains valid also in our present extreme case.

3. Parent-descendant combinations

By eliminating a type of one parent in a parents-descendant combination discussed in the preceding section, it then reduces to that of *a parent-descendant combination*. In fact, the probability of the latter is defined, for any $n\geqq 1$, by an equation

$$\mathfrak{k}_{t-1|n}(\alpha\beta;\, \xi\eta)\equiv\kappa_{(11;0)_{t-1}|n}(\alpha\beta;\, \xi\eta)=\sum \bar{A}_{ab}\mathfrak{e}_{t-1|n}(ab,\, \alpha\beta;\, \xi\eta).$$

In *case* $n=1$, its values are obtained as shown in the following lines:

$$\mathfrak{k}_{t-1|1}(ii;\; ii)=\frac{1}{2}(1+i)-(1-i)\boldsymbol{R}\left(\frac{2\sqrt{5}}{5}+i\frac{5-\sqrt{5}}{10}\right)\omega^t,$$

$$\mathfrak{k}_{t-1|1}(ii;\; ik)=k\boldsymbol{R}\left(\frac{4\sqrt{5}}{5}+i\frac{5-\sqrt{5}}{5}\right)\omega^t,$$

$$\mathfrak{k}_{t-1|1}(ii;\; kk)=\frac{1}{2}k-k\boldsymbol{R}\left(\frac{5+3\sqrt{5}}{10}-k\frac{5-\sqrt{5}}{10}\right)\omega^t,$$

$$\mathfrak{k}_{t-1|1}(ii;\; hk)=hk\boldsymbol{R}\frac{5-\sqrt{5}}{5}\omega^t;$$

$$\mathfrak{k}_{t-1|1}(ij;\; ii)=\frac{1}{4}(1+2i)-\boldsymbol{R}\left(\frac{5+3\sqrt{5}}{10}+i(1-i)\frac{5-\sqrt{5}}{10}\right)\omega^t,$$

$$\mathfrak{k}_{t-1|1}(ij;\; ij)=\boldsymbol{R}\left(\frac{5-\sqrt{5}}{10}+(i+j)\frac{2\sqrt{5}}{5}+ij\frac{5-\sqrt{5}}{5}\right)\omega^t,$$

$$\mathfrak{k}_{t-1|1}(ij;\; ik)=k\boldsymbol{R}\left(\frac{2\sqrt{5}}{5}+i\frac{5-\sqrt{5}}{5}\right)\omega^t,$$

$$\mathfrak{k}_{t-1|1}(ij;\; kk)=\frac{1}{2}k-k\boldsymbol{R}\left(\frac{5+3\sqrt{5}}{10}-k\frac{5-\sqrt{5}}{10}\right)\omega^t,$$

$$\mathfrak{k}_{t-1|1}(ij;\; hk)=hk\boldsymbol{R}\frac{5-\sqrt{5}}{5}\omega^t.$$

The *case* $n>1$, is treatable quite simply. In fact, by virtue of a relation $\mathfrak{e}_{t-1|n}=\varepsilon_n$ established in §2, we get readily the formula

$$\mathfrak{k}_{t-1|n}(\alpha\beta;\, \xi\eta)=\kappa_n(\alpha\beta;\, \xi\eta) \qquad \text{for } n>1;$$

cf. the rule stated in VI, §2.