# 148. *Uniform Convergence of Fourier Series. II*

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1. A. Zygmund has proved the following.

**Theorem 1.** *Let $0<\alpha<1$. If $f(x)$ is continuous and*

$$\omega(1/n)=o(1/n^{\alpha}),$$

*then the Fourier series of $f(x)$ is summable $(C, -\alpha)$ uniformly.*

This theorem was generalized by S. Izumi and T. Kawata [1] and S. Izumi [2]. We give another generalization of Theorem 1. In our theorem, the case where the modulus of continuity is of order $o(1/(\log n)^{\beta})$ is contained. (See Cor. 2.) The method of proof is analogous to [3]. (Cf. [4].)

2. **Theorem 2.** *If $f(x)$ is of class $\phi(n)$,*[1] *$\phi(n)$ being less than $n$, and is continuous with the modulus of continuity $\omega(\delta)$, then*[2]

$$|\sigma_n^{-\alpha}(x)-f(x)|\leqq C\left[\omega\left(\frac{1}{n}\right)^{1-\alpha}\left(\frac{n}{\phi(n)}\right)^{\alpha}+\frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\right],$$

*where $0<\alpha<1$ and $\sigma_n^{-\alpha}(x)$ is the nth Cesàro mean of the Fourier series of $f(x)$ of order $-\alpha$.*

**Proof.** We have

$$\sigma_n^{-\alpha}(x)-f(x)=\int_0^{\pi}\varphi_x(t)K_n^{-\alpha}(t)dt=\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right]\varphi_x(t)K_n^{-\alpha}(t)dt=I+J$$

say, where $K_n^{-\alpha}(t)$ is the Fejér kernel of order $-\alpha$, and $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$. It is known that

$$K_n^{-\alpha}(t)=\psi_n^{-\alpha}(t)+r_n^{-\alpha}(t) \tag{1}$$

where

$$\psi_n^{-\alpha}(t)=\cos\left(\left(n+\frac{1-\alpha}{2}\right)t-\frac{1-\alpha}{2}\pi\right)\Big/A_n^{-\alpha}\left(2\sin\frac{t}{2}\right)^{1-\alpha}, \tag{2}$$

$$r_n^{-\alpha}(t)=O(1/nt^2),\qquad |K_n^{-\alpha}(t)|\leqq Cn. \tag{3}$$

Then we get by (3)

$$I\leqq\int_0^{\pi/n}|\varphi_x(t)|\,|K_n^{-\alpha}(t)|dt\leqq Cn\int_0^{\pi/n}|\varphi_x(t)|\,dt\leqq Cn\omega\left(\frac{\pi}{n}\right)\int_0^{\pi/n}dt=C\omega\left(\frac{1}{n}\right).$$

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1) A function $f(x)$ is said to be of class $\phi(n)$ if $\phi(n)\uparrow\infty$ as $n\to\infty$ and

$$\int_a^b f(x+t)\cos nt\,dt=O(1/\phi(n))$$

uniformly for all $x$, $n$, $a$, $b$ with $b-a\leqq 2\pi$. (Cf. [4].) If $\omega(1/n)\leqq 1/\phi(n)$, then the condition becomes trivial, and hence we may suppose that $\omega(1/n)\geqq 1/\phi(n)$.

2) $C$ denotes an absolute constant, which need not be equal in each occurrence.

By (1), putting $\omega(1/n)=1/\theta(n)$,

$$J=\int_{\pi/n}^{\pi}\varphi_x(t)K_n^{-\alpha}(t)dt=\int_{\pi/n}^{\pi}\varphi_x(t)\psi_n^{-\alpha}(t)dt+\int_{\pi/n}^{\pi}\varphi_x(t)r_n^{-\alpha}(t)dt$$

$$=\left[\int_{\pi/n}^{a\theta(n)/\phi(n)}+\int_{a\theta(n)/\phi(n)}^{\pi}\right]\varphi_x(t)\psi_n^{-\alpha}(t)dt+\int_{\pi/n}^{\pi}\varphi_x(t)r_n^{-\alpha}(t)dt$$

$$=J_1+J_2+J_3,$$

say, where we take $a$ as the nearest number to 1 such that $an\theta(n)/\pi\phi(n)$ is an even integer. We have by (2)

$$J_1=\int_{\pi/n}^{a\theta(n)/\phi(n)}\varphi_x(t)\frac{\cos\left(\left(n+\frac{1-\alpha}{2}\right)t-\frac{1-\alpha}{2}\pi\right)}{A_n^{-\alpha}(2\sin t/2)^{1-\alpha}}dt$$

$$=\frac{1}{A_n^{-\alpha}}\left[\int_{\pi/n}^{a\theta(n)/\phi(n)}\varphi_x(t)\cos((1-\alpha)(t-\pi)/2)\frac{\cos nt}{(2\sin t/2)^{1-\alpha}}dt\right.$$

$$\left.+\int_{\pi/n}^{a\theta(n)/\phi(n)}\varphi_x(t)\sin((1-\alpha)(t-\pi)/2)\frac{\sin nt}{(2\sin t/2)^{1-\alpha}}dt\right]$$

$$=J_4+J_5,$$

say. Putting $\chi(t)=\varphi_x(t)\sin((1-\alpha)(t-\pi)/2)$ and $M=an\theta(n)/\pi\phi(n)$, then by the Salem method

$$J_5=\frac{1}{A_n^{-\alpha}}\int_{\pi/n}^{a\theta(n)/\phi(n)}\chi(t)\frac{\sin nt}{(2\sin t/2)^{1-\alpha}}dt$$

$$=\frac{1}{A_n^{-\alpha}}\int_{\pi/n}^{2\pi/n}\left\{\sum_{k=1}^{M}(-1)^k\frac{\chi(t+k\pi/n)}{(2\sin(t+2k\pi/n)/2)^{1-\alpha}}\right\}\sin nt\,dt$$

$$=\frac{1}{A_n^{-\alpha}}\int_{\pi/n}^{2\pi/n}\sum_{k=1}^{M/2}\left[\frac{\chi(t+2k\pi/n)-\chi(t+(2k+1)\pi/n)}{(2\sin(t+2k\pi/n)/2)^{1-\alpha}}\right]\sin nt\,dt$$

$$+\frac{1}{A_n^{-\alpha}}\int_{\pi/n}^{2\pi/n}\sum_{k=1}^{M/2}\chi(t+(2k+1)\pi/n)$$

$$\left[\frac{1}{(2\sin(t+2k\pi/n)/2)^{1-\alpha}}-\frac{1}{(2\sin(t+2(k+1)\pi/n)/2)^{1-\alpha}}\right]\sin nt\,dt$$

$$=J_6+J_7,$$

say, then

$$J_7\leqq Cn^{\alpha-1}\int_{\pi/n}^{2\pi/n}\sum_{k=1}^{M/2}\frac{\omega((2k+3)\pi/n)}{(t+2k\pi/n)^{2-\alpha}}dt\leqq C\omega\left(\frac{1}{n}\right)\cdot\left(\frac{n\theta(n)}{\phi(n)}\right)^{\alpha}.$$

On the other hand, since

$$|\chi(t+2k\pi/n)-\chi(t+(2k+1)\pi/n)|$$
$$\leqq|\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k+1)\pi/n)|+C/n,$$

we have

$$J_6\leqq Cn\int_{\pi/n}^{2\pi/n}\sum_{k=1}^{M/2}\frac{|\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k+1)\pi/n)|}{k^{1-\alpha}}dt$$

$$+\frac{C}{n}\sum_{k=1}^{M/2}\frac{1}{k^{1-\alpha}}$$

$$\leqq C\sum_{k=1}^{M/2}\frac{n}{k^{1-\alpha}}\int_{\pi/n}^{2\pi/n}|f(x+t+2k\pi/n)-f(x+t+(2k+1)\pi/n)|dt+\frac{CM^{\alpha}}{n}$$

$$\leqq C\left\{\omega\left(\frac{1}{n}\right)+\frac{1}{n}\right\}M^{\alpha}\leqq C\left\{\omega\left(\frac{1}{n}\right)+\frac{1}{n}\right\}\left(\frac{n\theta(n)}{\phi(n)}\right)^{\alpha}$$

$$\leqq C\left\{\omega\left(\frac{1}{n}\right)\left(\frac{n\theta(n)}{\phi(n)}\right)^{\alpha}+\left(\frac{n}{\phi(n)}\right)^{\alpha}\frac{1}{\theta(n)^{1-\alpha}}\right\}.$$

Thus

$$J_5\leqq C\omega\left(\frac{1}{n}\right)\left(\frac{n\theta(n)}{\phi(n)}\right)^{\alpha}+C\left(\frac{n}{\phi(n)}\right)^{\alpha}\frac{1}{\theta(n)^{1-\alpha}}=C\omega\left(\frac{1}{n}\right)^{1-\alpha}\left(\frac{n}{\phi(n)}\right)^{\alpha},$$

and $J_4$ has also the same estimate. Since $f(x)$ is of class $\phi(n)$,

$$J_2=\int_{a\theta(n)/\phi(n)}^{\pi}\varphi_x(t)\frac{\cos\left(\left(n+\frac{1-\alpha}{2}\right)t-\frac{1-\alpha}{2}\pi\right)}{A_n^{-\alpha}(2\sin t/2)^{1-\alpha}}dt$$

$$\leqq\frac{C}{A_n^{-\alpha}(a\theta(n)/\phi(n))^{1-\alpha}}\left|\int_{a\theta(n)/\phi(n)}^{\pi}\varphi_x(t)\cos\left(\left(n+\frac{1-\alpha}{2}\right)t-\frac{1-\alpha}{2}\pi\right)dt\right|$$

$$\leqq\frac{Cn^{\alpha}\phi(n)^{1-\alpha}}{\theta(n)^{1-\alpha}}\frac{1}{\phi(n)}=C\left(\frac{n}{\phi(n)}\right)^{\alpha}\frac{1}{\theta(n)^{1-\alpha}},$$

furthermore by (3)

$$J_3=\int_{\pi/n}^{\pi}\varphi_x(t)r_n^{-\alpha}(t)dt\leqq\frac{C}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt.$$

Thus we have

$$J\leqq C\left[\omega\left(\frac{1}{n}\right)^{1-\alpha}\left(\frac{n}{\phi(n)}\right)^{\alpha}+\frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\right],$$

which gives the required inequality with the estimation of $I$.

Taking $\phi(n)=1/\omega(1/n)$, we get

**Corollary 1.** $|\sigma_n^{-\alpha}(x)-f(x)|\leqq C\left[\omega\left(\frac{1}{n}\right)n^{\alpha}+\frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\right].$

3. **Theorem 3.** *If $f(x)$ is of class $\phi(n)$, $\phi(n)$ being less than $n$, and is continuous with modulus of continuity $\omega(\delta)$, and further $\omega\left(\frac{1}{n}\right)^{1-\alpha}\left(\frac{n}{\phi(n)}\right)^{\alpha}\to 0$ as $n\to\infty$ where $0<\alpha<1$, then the Fourier series of $f(x)$ is summable $(C,-\alpha)$ uniformly.*

We can easily prove by Theorem 2.

Furthermore we get Theorem 1, taking $\omega(1/n)=o(1/n^{\alpha})$ in Corollary 1.

**Corollary 2.** *Let $0<\alpha<1$. If $f(x)$ is continuous, $\omega(1/n)=(1/n^{\alpha})$ and*

$$\int_a^b f(x+t)\cos nt\,dt=o(1/n^{\alpha})\ \textit{unif. in}\ x,n,a,b\quad(b-a\leqq 2\pi)$$

*then the Fourier series of $f(x)$ is summable $(C,-\alpha)$ uniformly.*

For the proof it is sufficient to take $1/\phi(n)=o(1/n^{\alpha})$ in Theorem 3.

**Corollary 3.** *Let* $0<\alpha<1$ *and* $\beta>0$. *If* $f(x)$ *is continuous,* $\omega(1/n)=o(1/(\log n)^{\beta})$ *and*

$$\int_a^{\beta} f(x+t)\cos nt\,dt=O((\log n)^{\beta/\alpha-\beta}/n)\ \textit{unif. in } x, n, a, b\ (b-a\leqq 2\pi),$$

*then the Fourier series of* $f(x)$ *is summable* $(C, -\alpha)$ *uniformly.*

For the proof it is sufficient to take $\phi(n)=n/(\log n)^{\beta/\alpha-\beta}$ in Theorem 3.

## References


[1] S. Izumi and T. Kawata: Notes on Fourier series IX, Tôhoku Mathematical Journal, **46** (1939).

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[3] M. Satô: Uniform convergence of Fourier series, Proc. Japan Acad., **30** (1954).

[4] J. P. Nash: Uniform convergence of Fourier series, Rice Institute Pamphlet (1953).