160. On the Cell Structures of $SU(n)$ and $Sp(n)$

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The Betti numbers of the classical groups (the special orthogonal group $SO(n)$, the special unitary group $SU(n)$, and the symplectic group $Sp(n)$) were determined by the various methods.[1] Recently, by making use of the spectral sequences for the fibre spaces $SO(n)/SO(n-1)=S^{n-1}$, $SU(n)/SU(n-1)=S^{2n-1}$, and $Sp(n)/Sp(n-1)=S^{4n-1}$, A. Borel[2] has computed the integral homology groups of these groups. As to $SO(n)$, J. H. C. Whitehead[3] has determined its cell structure as a cell complex. Those cells were closely connected with real projective space P. C. E. Miller[4] has computed the homological and the cohomological structures by making use of the above cell structure.

In this paper we shall determine the cell structures of $SU(n)$ and $Sp(n)$ as cell complexes. Those cells are closely connected with the first suspended space $E(M)$ of the complex projective space M and the third suspended space $E^3(Q)$ of the quaternion projective space Q respectively. The above considerations also give the cellular decompositions of the complex Stiefel manifold $W_{n,m}=SU(n)/SU(n-m)$ and the quaternion Stiefel manifold $X_{n,m}=Sp(n)/Sp(n-m)$.

Using this cell structure, the homology groups and the cohomology groups are computed very easily. If we want to calculate the cup product, the Pontrjagin product, and the Steenrod's square operations etc., we shall be able to attain the aim with some more preparations. The full details will appear in the Journal of the Institute of Polytechnics, Osaka City University.

1. Let C^n be a vector space of dimension n over the field of complex numbers, and e_i be the element of C^n whose i-th coordinate is 1 and whose other coordinates are 0. We embed C^n in C^{n+1} as a subspace whose first coordinate is 0. Let S^{2n-1} be the unit sphere of C^n, then the embedding $C^n \subset C^{n+1}$ gives rise to an embedding $S^{2n-1} \subset S^{2n+1}$.

1) J. K. Koszul: Homologie et cohomologie de algèbres de Lie, Bull. Soc. Math. France, **78** (1950). H. Samelson: Beiträge zur Topologie der Gruppen Mannigfaltigkeiten, Ann. Math., **42** (1941).

2) A. Borel: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. Math., **57** (1953).

3) J. H. C. Whitehead: On the groups $\Pi_r(V_{n,m})$ and sphere bundles, Proc. London Math. Soc., **48** (1954).

4) C. E. Miller: The topology of rotation groups, Ann. Math., **57** (1953).

Let $SU(n)$ be the group of all special unitary linear transformations of C^n. In matrix notation, (n, n) matrix A with complex coefficients is special unitary if and only if $AA^*=E$[5] and $det\, A=1$. $SU(n)$ may be regarded as a subgroup of $SU(n+1)$ by extending a matrix A of $SU(n)$ to $SU(n+1)$ by requirement that $Ae_1=e_1$.

Set $p(A)=Ae_1$ for $A \in SU(n)$. Then by the map p, $SU(n)$ operates on S^{2n-1} transitively and its isotoropy group is $SU(n-1)$. Hence we have $SU(n)/SU(n-1)=S^{2n-1}$.

2. Let M_n be the $2n$-dimensional projective space. If a point x of M_n has a representative $x=[x_1, x_2, \dots, x_{n+1}]$, then the other representatives are $x=[ax_1, ax_2, \dots, ax_{n+1}]$ where a is any non zero complex number. In the following, we shall choose a representative such that $|x_1|^2+|x_2|^2+\cdots+|x_{n+1}|^2=1$. We can regard M_n as a subspace of M_{n+1} whose first coordinate is 0.

Let $E(M_n)$ be the suspended space of M_n. This definition is the following. Let I be the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\left(\frac{\pi}{2}\right)_*$, $\left(-\frac{\pi}{2}\right)_*$ be two different points which are not in M_n. Then, $E(M_n)$ is the space formed from $M_n \times I$ by contracting $M_n \times \frac{\pi}{2}$ and $M_n \times \left(-\frac{\pi}{2}\right)$ respectively to $\left(\frac{\pi}{2}\right)_*$ and $\left(-\frac{\pi}{2}\right)_*$. Thus a point of $E(M_n)$ has the coordinates (x, θ), where $x \in M_n$, $\theta \in I$. Especially the coordinates of $\left(\frac{\pi}{2}\right)_*$ and $\left(-\frac{\pi}{2}\right)_*$ are respectively $\left(x, \frac{\pi}{2}\right)$ and $\left(x, -\frac{\pi}{2}\right)$, where x is an arbitrary point of M_n.

3. Define a map $f: E(M_{n-1}) \to SU(n)$ by $f(x, \theta)=U=VW$, where $x=[x_1, x_2, \dots, x_n] \in M_{n-1}$ such that $|x_1|^2+|x_2|^2+\cdots+|x_n|^2=1$, $\theta \in I$,

$$V=\exp\theta\,(\sqrt{-1}\,\theta)E-2\cos\theta\begin{pmatrix} |x|^2 & \bar{x}_2x_1 & \dots & \bar{x}_nx_1 \\ \bar{x}_1x_2 & |x_2|^2 & \dots & \bar{x}_nx_2 \\ \dots & \dots & \dots & \dots \\ \bar{x}_1x_n & \bar{x}_2x_n & \dots & |x_n|^2 \end{pmatrix},$$

and
$$W=\begin{pmatrix} \exp(-\sqrt{-1}\,\theta) & & & & \\ & \exp(-\sqrt{-1}\,\theta) & & & \\ & & \ddots & & \\ & & & \exp(-\sqrt{-1}\,\theta) & \\ & & & & -\exp(\sqrt{-1}\,\theta) \end{pmatrix}.$$

U does not depend on the choise of representatives of x, and if $\theta=\pm\frac{\pi}{2}$, U also does not depend on x. Therefore, f is well defined. It will be easily verified that U is special unitary. We shall call

5) A^* is the transposed and conjugate matrix of A. E is the unit matrix.

f the characteristic map of $E(M_{n-1})$ into $SU(n)$.

4. Define a map $\xi: E(M_{n-1}) \to S^{2n-1}$ by $\xi = pf$, then ξ *maps* $E(M_{n-2})$ *to a point* e_1 *of* S^{2n-1} *and* $\mathcal{E}^{2n-1} = E(M_{n-1}) - E(M_{n-2})$ *homeomorphically onto* $S^{2n-1} - e_1$.

In fact, it is obvious that ξ maps $E(M_{n-2})$ to e_1. Given any point $(\alpha + \beta\sqrt{-1}, a_2, a_3, \ldots, a_n)$ of $S^{2n-1} - e_1$, where $\alpha \neq 1, \beta$ are real numbers and $a_1, a_3, \ldots, a_n$ are complex numbers, it is sufficient to show the following equations can be solved continuously:

$$\begin{cases} 1 - 2\exp(-\sqrt{-1}\,\theta)\cos\theta\,|x_1|^2 = \alpha + \beta\sqrt{-1} \\ -2\exp(-\sqrt{-1}\,\theta)\cos\theta\,\bar{x}_1 x_2 = a_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -2\exp(-\sqrt{-1}\,\theta)\cos\theta\,\bar{x}_1 x_n = a_n. \end{cases}$$

From the first equation, we have

$$x_1 = \frac{\sqrt{(1-\alpha)^2 + \beta^2}}{\sqrt{1-\alpha}} \exp(\sqrt{-1}\,\varphi), \quad \sin\theta = \frac{\beta}{\sqrt{(1-\alpha)^2 + \beta^2}},$$

where φ is an arbitrary real number. Thus x_1 and θ are determined. From the other equations, $x_2, \ldots, x_n$ can be determined. Thus $x = [x_1, x_2, \ldots, x_n]$ has determined uniquely as a point of the projective space M_{n-1}.

5. In the preceding section, we saw that f mapped $\mathcal{E}^{2k-1}$ homeomorphically into $SU(k) \subset SU(n)$ for $n \geqq k \geqq 2$. Set $e^{2k-1} = f(\mathcal{E}^{2k-1})$ and we shall call it $2k-1$ dimensional *primitive cell* of $SU(n)$. Thus we have $3, 5, 7, \ldots, 2n-1$ dimensional $n-1$ primitive cells of $SU(n)$.

For $n \geqq k_1 > k_2 \cdots > k_j \geqq 2$, extend f to a map $\bar{f}: E(M_{k_1-1}) \times E(M_{k_2-1}) \times \cdots \times E(M_{k_j-1}) \to SU(n)$ by $\bar{f}((x, \theta_1) \times (y, \theta_2) \times \cdots \times (z, \theta_j)) = f(x, \theta_1) f(y, \theta_2) \ldots f(z, \theta_j)$. Put $e^{2k_1-1, 2k_2-1, \ldots, 2k_j-1} = \bar{f}(\mathcal{E}^{2k_1-1} \times \mathcal{E}^{2k_2-1} \times \cdots \times \mathcal{E}^{2k_j-1})$.

First of all, we shall show that $SU(n)$ is the union of cells $e^0 = E, e^{2k-1}$ and $e^{2k_1-1, 2k_2-1, \ldots, 2k_j-1}$, where $n \geqq k_1 > k_2 > \cdots > k_j \geqq 2$. Since $SU(1) = E$, we shall assume that the above assertion is true for $SU(m)$ where $m < n$. If $A \in SU(n)$ but $A \notin SU(n-1)$, then we can choose a point $(x, \theta) \in \mathcal{E}^{2n-1}$ uniquely such that $\xi(x, \theta) = p(A)$. Put $U = f(x, \theta)$, then $U^*A \in SU(n-1)$. Hence U^*A belongs to a certain cell $e^{2k_1-1, 2k_2-1, \ldots, 2k_j-1}$, where $n-1 \geqq k_1 > k_2 > \cdots > k_j \geqq 2$, of $SU(n-1)$ by the inudction. Therefore, A belongs to a cell $e^{2n-1, 2k_1-1, \ldots, 2k_j-1}$.

Next, we shall show that $\bar{f}$ maps $\mathcal{E}^{2k_1-1} \times \mathcal{E}^{2k_2-1} \times \cdots \times \mathcal{E}^{2k_j-1}$ homeomorphically onto $e^{2k_1-1, 2k_2-1, \ldots, 2k_j-1}$ and these cells are disjoint to each other. In fact, if $U_1 U_2 \ldots U_s = V_1 V_2 \ldots V_t$, where $U_m \in e^{2k_m-1}$ and if $m < m'$ then $k_m > k_{m'}$ and V_l is the similar one, $p(U_1 U_2 \ldots U_s) = p(V_1 V_2 \ldots V_t)$. Since $p(U_1 U_2 \ldots U_s) = p(U_1)$, $p(U_1) = p(V_1)$. Since ξ is homeomorphic, it follows $U_1 = V_1$. Hence, $U_2 U_3 \ldots U_s = V_2 V_3 \ldots V_t$.

Similarly $U_2=V_2$ and so on. Consequently $s=t$. The fact that $\bar{f}$ is a homeomorphism is obvious from that of ξ.

Thus we have the following

Theorem. *The special unitary group $SU(n)$ is a cell complex composed of 2^{n-1} cells e^0, $e^{2k_1-1,2k_2-1,\dots,2k_j-1}$, where $n\geqq k_1>k_2>\cdots>k_j\geqq 2$. The dimension of $e^{2k_1-1,2k_2-1,\dots,2k_j-1}$ is $(2k_1-1)+(2k_2-1)+\cdots+(2k_j-1)$. Especially e^{2k-1} called $2k-1$ dimensional primitive cell of $SU(n)$ is obtained as the image of the interior of the suspended space $E(M_{k-1})$ of $2k-2$ dimensional complex projective space M_{k-1} by the characteristic map $f: E(M_{k-1})\to SU(k)\subset SU(n)$.*

With respect to the above cell structure of $SU(n)$, the boundary homomorphisms are trivial in all dimensions. Hence we can compute the homology groups and the cohomology groups very easily. In fact, *the torsion groups are zero in all dimensions*, and the Betti number for m dimension is the number of the cells whose dimensions are m. Therefore, the Poincaré polynomial of $SU(n)$ is

$$P_{SU(n)}(t)=(1+t^3)(1+t^5)\dots(1+t^{2n-1}).$$

6. Instead of the field of complex numbers, if we take the field of quaternion numbers, the considerations of §§1–5 are also extensible to the case of the symplectic group $Sp(n)$.

Let Ω^n be a vector space of dimension n over the field of quaternion numbers, and embed Ω^n in Ω^{n+1} as a subspace whose first coordinate in 0. Let S^{4n-1} be the unit sphere in Ω^n, then $\Omega^n\subset\Omega^{n+1}$ gives rise to $S^{4n-1}\subset S^{4n+3}$.

Let $Sp(n)$ be the group of all symplectic linear transformations of Ω^n. Namely, in matrix notation, (n, n) matrix A with quaternion coefficients is symplectic if and only if $AA^*=E$. $Sp(n)$ may be regarded as a subgroup of $Sp(n+1)$ by extending an element A of $Sp(n)$ to $Sp(n+1)$ by the requirement that $Ae_1=e_1$. Set $p(A)=Ae_1$ for $A\in SU(n)$. Then by the map p, $Sp(n)$ operates on S^{4n-1} transitively and its isotoropy group is $Sp(n-1)$. Hence we have $Sp(n)/Sp(n-1)=S^{4n-1}$.

7. Let Q_n be the $4n$ dimensional quaternion projective space. If a point x of Q_n has a representative $x=[x_1, x_2,\dots, x_{n+1}]$, then the other representatives are $x=[ax_1, ax_2,\dots, ax_{n+1}]$, where a is any non zero quaternion number. We can regard Q_n as a subspace of Q_{n+1} whose first coordinate is 0.

Let $E^3(Q_n)$ be the third suspended space Q_n. Its definition is the following. Let E^3 be the closed cell consisting of all pure imaginary quaternion numbers whose norm $\leqq 1$, S_1^2 be its boundary and S_{1*}^2 be a 2-dimensional sphere which is not in Q_n. Choose a homeomorphism η of S_1^2 to S_{1*}^2 and put $\eta(q)=q_*$. Then $E^3(Q_n)$ is the space formed from $Q_n\times E^3$ by contracting $Q_n\times q$ to q_* for each

$q \in S_1^2$. Thus a point $E^3(Q_n)$ has the coordinates (x, q), where $x \in Q_n$, $q \in E^3$. Especially a point of S_{1*}^2 has coordinates (x, q), where x is an arbitrary point of Q_n and $q \in S_1^2$.

8. Define a map $f: E^3(Q_{n-1}) \to Sp(n)$ by $f(x, q) = U$, where $x = [x_1, x_2, \dots, x_n] \in Q_{n-1}$ such that $|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = 1$, $q \in E^3$,

$$U = E - 2\sqrt{1-|q|^2}(-q + \sqrt{1-|q|^2})\begin{pmatrix} |x_1|^2 & \bar{x}_2 x_1 & \dots & \bar{x}_n x_1 \\ \bar{x}_1 x_2 & |x_2|^2 & \dots & \bar{x}_n x_2 \\ \dots & \dots & \dots & \dots \\ \bar{x}_1 x_n & \bar{x}_2 x_n & \dots & |x_n|^2 \end{pmatrix}.$$

U does not depend on the choise of representatives of x, and if $q \in S_{1*}^2$, U also does not depend on x. Therefore, f is well defined. It will be easily verified that U is symplectic.

9. Define a map $\xi: E^3(Q_{n-1}) \to S^{4n-1}$ by $\xi = pf$, then ξ *maps* $\mathcal{E}^{4n-1} = E^3(Q_{n-1}) - E^3(Q_{n-2})$ *homeomorphically onto* $S^{4n-1} - e_1$ *and contracts* $E^3(Q_{n-2})$ *to a point* e_1.

10. Set $e^{4k-1} = f(\mathcal{E}^{4k-1})$ and we shall call it $4k-1$ dimensional *primitive cell* of $Sp(n)$. Thus we have $3, 7, 11, \dots, 4n-1$ dimensional n primitive cells of $Sp(n)$.

For $n \geqq k_1 > k_2 > \cdots > k_j \geqq 1$, extend f to a map $\bar{f}: E^3(Q_{k_1-1}) \times E^3(Q_{k_2-1}) \times \cdots \times E^3(Q_{k_j-1}) \to Sp(n)$ by $\bar{f}((x, \theta_1) \times (y, \theta_2) \times \cdots (z, \theta_j)) = f(x, \theta_1) f(g, \theta_2) \dots f(z, \theta_j)$. Put $e^{4k_1-1, 4k_2-1, \dots, 4k_j-1} = f(\mathcal{E}^{4k_1-1} \times \mathcal{E}^{4k_2-1} \times \cdots \times \mathcal{E}^{4k_j-1})$. Then we have the following theorem as similar as $SU(n)$.

Theorem. *The symplectic group $Sp(n)$ is a cell complex composed of 2^n cells e^0, $e^{4k_1-1, 4k_2-1, \dots, 4k_j-1}$, where $n \geqq k_1 > k_2 > \cdots > k_j \geqq 1$. The dimension of $e^{4k_1-1, 4k_2-1, \dots, 4k_j-1}$ is $(4k_1-1) + (4k_2-1) + \cdots + (4k_j-1)$. Especially e^{4k-1} called $4k-1$ dimensional primitive cell of $Sp(n)$ is obtained as the image of the interior of the third suspended space $E^3(Q_{k-1})$ of $4k-4$ dimensional quaternion projective space Q_{k-1} by the characteristic map $f: E^3(Q_{k-1}) \to Sp(k) \subset Sp(n)$.*

With respect to this cell structure of $Sp(n)$, the boundary homomorphisms are trivial in all dimensions. So that *the torsion groups are zero in all dimensions*, and the Poincaré polynomial of $Sp(n)$ is

$$P_{Sp(n)}(t) = (1+t^3)(1+t^7)\dots(1+t^{4n-1}).$$