93. On Solutions of a Partially Differential Equation with a Parameter

By Taira SHIROTA
Osaka University

(Comm. by K. KUNUGI, M.J.A., June 12, 1956)

Let $P\left(\frac{\partial}{\partial x}, \lambda\right)$ be a polynomial of derivations $\sum_{|p|\leq m} a_p(\lambda)\frac{\partial^p}{\partial x^p}$ defined in R^n and with a parameter λ, where $a_p(\lambda)$ are complex valued continuous functions on a separable and locally compact space Λ and where the degrees of polynomials $P(\xi, \lambda)$ are independent of $\lambda \in \Lambda$. Then we consider the existence of distribution solution, which is continuous with respect to $\lambda \in \Lambda$, of the partially differential equation

$$P\left(\frac{\partial}{\partial x}, \lambda\right)S_x(\lambda)=T_x(\lambda), \tag{1}$$

where $T_x(\lambda)$ is a given continuous function on Λ into a distribution space.

In the special case in this direction where Λ consists of a point, many interesting results are obtained by B. Malgrange, L. Hörmander and L. Ehrenpreis.[1] Furthermore recently F. Tréves[2] considered the case where $T_x(\lambda)=\delta$. Here we prove more general theorems applying considerations of these author's.

Theorem 1. *For any continuous function $T_x(\lambda)$ on Λ into $\mathfrak{D}'_x$ there is a solution $S_x(\lambda)$ of the equation (1) where $S_x(\lambda)$ is a continuous function on Λ into $\mathfrak{D}'_x$ and where $S_x(\lambda)=0$ whenever $T_x(\lambda)=0$.*

Theorem 2. *Under the same assumption of Theorem 1, if $S_x(\lambda)$ is a continuous solution such that*

$$P\left(\frac{\partial}{\partial x}, \lambda\right)S_x(\lambda)=T_x(\lambda) \qquad \text{for } \lambda \in \Lambda_0 \tag{2}$$

where Λ_0 is a closed subspace of Λ, then there is a continuous solution $S'_x(\lambda)$ of (1) defined over Λ such that

$$S'_x(\lambda)=S(\lambda) \qquad \textit{for } \lambda \in \Lambda_0. \tag{3}$$

Furthermore we may replace $\mathfrak{D}'_x$ by $\mathcal{E}_x$, that is, we can prove the following

1) B. Malgrange: Equations aux dérivées partielles à coefficients constants. I, II, C. R. Acad. Sci., Paris, **237** (1953), **238** (1954). L. Hörmander: On the theory of general partial differential operators, Acta Math., **94** (1955). L. Ehrenpreis: The division problem for distributions, Proc. Nat. Acad. Sci., **41** (1955).

2) F. Tréves: Solution élémentaire d'équations aux dérivées partielles dépendant d'un paramètre, C. R. Acad. Sci., Paris, **242** (1956).

Theorem 3. *For any continuous function $f(x,\lambda)$ on Λ into $\mathcal{E}_x$, there is a solution $g(x,\lambda)$ such that*

$$P\left(\frac{\partial}{\partial x},\lambda\right)g(x,\lambda)=f(x,\lambda) \tag{4}$$

where $g(x,\lambda)$ is a continuous function on Λ into $\mathcal{E}_x$.

Moreover if $g(x,\lambda)$ is a continuous solution such that

$$P\left(\frac{\partial}{\partial x},\lambda\right)g(x,\lambda)=f(x,\lambda) \qquad \text{for } \lambda\in\Lambda_0$$

where Λ_0 is a closed subspace of Λ, then we can find a continuous extension $g'(x,\lambda)$ of the equation (4) over all $\lambda\in\Lambda$.

Theorem 2 immediately follows from Theorem 1. To show this, let $S(\lambda)$ be a continuous solution satisfying (2). Then by Theorem 1, there is a continuous solution $S'(\lambda)$ of (1). Let $S''(\lambda)$ be a continuous extension[3] of $S(\lambda)-S'(\lambda)$ over Λ. Since

$$P\left(\frac{\partial}{\partial x},\lambda\right)(S(\lambda)-S'(\lambda))=0 \qquad \text{for } \lambda\in\Lambda_0,$$

by Theorem 1, there is a continuous solution $S'''(\lambda)$ such that

$$P\left(\frac{\partial}{\partial x},\lambda\right)S'''(\lambda)=P\left(\frac{\partial}{\partial x},\lambda\right)S''(\lambda),$$

and

$$S'''(\lambda)=0 \qquad \text{for } \lambda\in\Lambda_0.$$

Let $S(\lambda)$ be $S'(\lambda)+S''(\lambda)-S'''(\lambda)$. Then $S(\lambda)$ is a solution of (1) which satisfies the equation (3).

The proof of Theorem 1.[4] 1. First of all we show that we have only to prove the case where Λ is compact and

$$P(\xi,\lambda)=\xi_1^m+Q(\xi_1,\xi_2,\cdots,\xi_n,\lambda) \tag{5}$$

with the $\deg_{\xi_1}Q(\xi,\lambda)<m$. For assume that in this case our theorem is proved. Then since Λ is separable and locally compact, there is a locally finite open covering $\{U_\alpha\}$ of Λ such that U_α is compact and such that for any α $P_0(\xi_\alpha,\lambda)\neq 0$ for any $\lambda\in U_\alpha$ and for a fixed point $\xi_\alpha\in R^n$, where P_0 is the principal part of P. Hence for any α by a coordinate transformation on R^n $P(\xi,\lambda)$ assumes the above-mentioned form. Furthermore let $\{f_\alpha\}$ be a partition of 1 with respect to $\{U_\alpha\}$. Then by the assumption there is for any α a continuous solution $S_\alpha(\lambda)$ such that

$$P(\xi,\lambda)S_\alpha(\lambda)=f_\alpha(\lambda)T_\alpha(\lambda), \quad \lambda\in U_\alpha,$$

and

$S(\lambda)=0$ for $\lambda\in$ some neighbourhood of the boundary of U_α. Thus setting $S_\alpha(\lambda)=0$ for $\lambda\notin U_\alpha$, we obtain the desired solution $\sum_\alpha S_\alpha(\lambda)$.

3) J. Dugundji: An extension of Tietze's theorem, Pacific J. Math., **1** (1951).

4) The proof of Theorem 3 may be accomplished by using the duality of $\mathcal{E}$ and $\mathcal{E}'$ and is similar to, but simpler than the proof of Theorem 1. Therefore we shall omit the proof.

2. From now on we assume that Λ is compact and $P(\xi,\lambda)$ assumes the form described in (5). Now we decompose T_λ as follows.

Let $\{d_t \mid t=1,2,\cdots\}$ be a monotone increasing sequence such that for open sets $O_0=\{x \mid \|x\|<1\}$ and $O_t=\{x \mid \|x\|<t+1 \ \& \ x_i>d_t,\ i=1,2,\cdots,n\}$, $\sum_{i=0}^{\infty} O_i$ contains a proper open cone with vertex O and center $\{x \mid x_i=x_j>0,\ i,j=1,2,\cdots,n\}$. Then there is a finite number of regular transformations $\Sigma_1=I, \Sigma_2,\cdots,\Sigma_l$ such that $\underset{\substack{i=1,2,\cdots\\ j=1,2,\cdots,l}}{U} \Sigma_j(O_i)+O_0=R^n$. Let $\{f_{ji}\}$ be a partition of 1 with respect to the covering and let $T_{\lambda ji}=f_{ji}T_\lambda$. Furthermore let $T_{\lambda 1}=\sum_{i=0}^{\infty} T_{\lambda 1i}$ and let $T_{\lambda j}=\sum_{i=1}^{\infty} T_{\lambda ji}$. Then we have only to prove the theorem for the case where the right hand side of (1) is $T_{\lambda 1}$, that is, the case where the carrier of T_λ is contained in $\bigcup_{i=0}^{\infty} O_i$. For let $T'_{\lambda j}(\varphi)=T_{\lambda j}(\varphi(\Sigma_j^{-1}\cdot))$. Then the carrier of $T'_{\lambda j}$ is contained in $\bigcup_{i=1}^{\infty} O_i$. Thus if there are solutions $S'_{\lambda j}$ such that $P\left(\frac{\partial}{\partial x},\lambda\right)S'_{\lambda j}=T'_{\lambda j}$, setting $S_{\lambda j}(\varphi)=S'_{\lambda j}(\varphi(\Sigma_j\cdot))$, we obtain the desired solution $\sum_j S_{\lambda j}$.

3.[5] From now on we restrict our argument to the case where the right hand side of our equation (1) is $T_\lambda=\sum_{t=0}^{\infty} T_{\lambda t}$ with carriers (of $T_{\lambda t}\subset O_t$). Then $T_{\lambda t}$ are continuous functions on Λ into $\mathcal{E}'_x(O_t)$. Since $\{T_{\lambda t} \mid \lambda\in\Lambda\}$ is compact in $\mathcal{E}'_x(O_t)$, for any t there are an integer s_t and a number k_t such that

$$T_{\lambda t}=P_{\lambda t}\left(\frac{\partial}{\partial x}\right)f_{\lambda t}$$

where $\deg P_{\lambda t}\leqq s_t$, $\int |f_{\lambda t}|\,dx\leqq k_t$ and the carrier of $f_{\lambda t}\subset O_t$.

Now let r be a positive number and let $\varphi_r(\xi)$ be the unique positive solution of the equation $2r\eta=\log(\xi^2+\eta^2)$ for real $\xi(|\xi|\geqq\sqrt{e})$. Then we denote by $\gamma(r)$ the curve in C' defined by

$$\begin{aligned}&\eta=\varphi_r(\xi) && \text{for } |\xi|>\sqrt{e}\\ &\eta=\varphi_r(\sqrt{e}) && \text{for } |\xi|\leqq\sqrt{e}\ \ (\varphi_0(\xi)=0)\end{aligned} \tag{6}$$

where $\zeta=\xi+i\eta$. Furthermore let $r_t=\frac{d_t}{b_t}$, where b_t is chosen sufficiently large such that

(i) $\{r_t\}$ is monotone decreasing and $r_t\leqq a$ for sufficiently small positive number a, and (7)

(ii) $\sum_{t=1}^{\infty} k_t(\exp(-\varphi_{r_t}(\sqrt{e}))+r_t)<\infty$.

5) The idea of the step 3 is the same as one of Ehrenpreis, but I obtained this natural idea independently of him.

Then for any $\varphi \in \mathfrak{D}_x$,

$$T_\lambda(\varphi)=\sum_{t=0}^{\infty} T_{\lambda t}(\varphi)=\sum_{t=0}^{\infty}\int_{R^n} \mathfrak{F}^{-1}(T_{\lambda t})(\xi)\mathfrak{F}(\varphi)(\xi)d\xi$$

$$=\int_{R^n} \mathfrak{F}^{-1}(T_{\lambda 0})(\xi)\mathfrak{F}(\varphi)(\xi)d\xi+\sum_{t=1}^{\infty}\int_{\gamma(r_t)^n} \mathfrak{F}^{-1}(T_{\lambda t})(\zeta)\mathfrak{F}(\varphi)(\zeta)d(\zeta).$$

For $\mathfrak{F}^{-1}(T_{\lambda t})(\zeta)\mathfrak{F}(\varphi)(\zeta)$ is an entire analytic function on C^n,

$$|\mathfrak{F}^{-1}(T_{\lambda t})(\zeta)| \leqq k_t(1+|\zeta|^{s_t}) \exp(-d_t|\eta|) \tag{8}$$

and

$$|\mathfrak{F}(\varphi)(\zeta)| \leqq M(\varphi, s)(1+|\zeta|)^{-s} \exp(l|\eta|)$$

for any ζ with $\eta_i \geqq 0$ $(i=1, 2, \cdots, n)$ and for any integer $s>0$. Hence it is a consequence of Cauchy's integral theorem.

4. Let q be a continuous function Λ into $R^{l'}$ such that $q(\lambda)=\{a_p(\lambda)\}$ and let $P'(\xi, q(\lambda))=P(\xi, \lambda)$. Then $P'(\xi, \lambda')$ may be extended over a relatively compact open set Λ' of $R^{l'}$. For any j $(=1, 2, \cdots, 3m-1)$ let Q_{tj} be the neighbourhood of $\gamma(r_t)+2ji$ with radius 1 and let $U_{tj}=\{(\zeta^*, \lambda) \mid P'(\zeta_1, \zeta^*, \lambda)=0 \text{ implies } \zeta_1 \notin \overline{Q}_{tj}\}$ where $\zeta^* \in \gamma(r_t)^{n-1}$ and $\lambda' \in \Lambda'$. Then from the form (5) of $P'(\xi, \lambda')$, $\{U_{tj} \mid j=1, 2, \cdots, 3m-2\}$ is an open covering of $\gamma(r_t)^{n-1}\times\Lambda'$. Now let $p_t(\zeta)$ be the projection $(p_t(\zeta)=\xi)$ from $\gamma(r_t)^{n-1}$ onto R^{n-1} and let $p'(\zeta, \lambda')=(p(\zeta), \lambda')$. Then $\{p'(U_{tj})\}$ is an open covering of $R^{n-1}\times\Lambda'$. Accordingly there is a locally finite partition $\{U_{tji} \mid j=1, 2, \cdots, 3m-2,\ i=1, 2, 3, \cdots\}$ of $R^{n-1}\times\Lambda'$ such that any U_{tji} is an open rectangle whose sides are not parallel to R^{n-1} and such that $U_{tji}\subset P'(U_j)$. Let $U_{tji}(\lambda')$ be $\{\xi \mid (\xi, \lambda')\in U_{tji}\}$, then $\operatorname{mes}(U_{tji}(\lambda_1')\,\Delta\, U_{tji}(\lambda_2'))\to 0$ when $\lambda_1'\to\lambda_2'$ and $\operatorname{mes}(R^{n-1}-\bigcup_{ji} U_{tji}(\lambda'))=0$ for any $\lambda'\in\Lambda'$. Furthermore let $p_t^{-1}\Big(\sum_{i=1}^{\infty} U_{tji}(\lambda')\Big)=W_{tj}(\lambda')$.

Finally let $S_{\lambda tj}(\varphi)$ be

$$\int_{W_{tj}(q(\lambda))} d\zeta^* \int_{\gamma(r_t)} \frac{\mathfrak{F}^{-1}(T_{\lambda t})(\zeta_1+2ji, \zeta^*)\mathfrak{F}(\varphi)(\zeta_1+2ji, \zeta^*)}{P(\zeta_1+2ji, \zeta^*, \lambda)}d\zeta_1$$

and let $S_\lambda(\varphi)=\sum_{t=0}^{\infty}\sum_{j=1}^{3m-2} S_{\lambda tj}(\varphi)$. Then $S_\lambda(\varphi)$ is an absolutely convergent series and uniformly bounded for any $\varphi\in B\subset\mathfrak{D}_x$ where B is any bounded set in $\mathfrak{D}_x$. For $P(\zeta_1+2ji, \zeta^*, \lambda)\geqq 1$ for any $(\zeta_1, \zeta^*, \lambda)\in\gamma(r_t)\times W_{tj}(q(\lambda))$, therefore we may consider the series

$$\sum_{tj}\int_{\gamma(r_t)^{n-1}} d\zeta^* \int_{\gamma(r_t)} |\mathfrak{F}^{-1}(T_{\lambda t})\mathfrak{F}(\varphi)|(\zeta_1+2ji, \zeta^*)d\zeta_1.$$

But from (6), (7), (8) and

$$|\mathfrak{F}\varphi(\zeta)| \leqq M(B, s)(1+|\zeta|)^{-s}\exp(l|\eta|) \qquad \text{for any } \varphi\in B,$$

where s is any positive integer and where l and $M(B, s)$ are constants independent of φ, the series converges uniformly to a finite value for any $\lambda\in\Lambda$ and for any $\varphi\in B$. Thus we see that S_λ is a distribution.

Furthermore S_λ is continuous from Λ into $\mathfrak{D}'_x$. For $\mathfrak{F}^{-1}(T_{\lambda t})(\zeta)$ is a continuous function on Λ into $C(\{\zeta \mid \|\zeta\| \leqq k\})$ for any $k>0$. Hence the continuity of S_λ follows from the above-mentioned construction of $\{U_{tj}(q(\lambda))\}$. Moreover from the step 3 it shows that

$$P\left(\frac{1}{-2\pi i}\frac{\partial}{\partial x}, \lambda\right)S(\lambda)=T(\lambda)$$

and that $S(\lambda)=0$ whenever $T(\lambda)=0$.