98. On the Group Structure of Boolean Lattices

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Boolean lattices can be considered as groups with the composition $\triangle$ (symmetric difference: $a\triangle b=(a\frown b^c)\smile(a^c\frown b)$) and the unit 0. We treated in [1] (or $[1]_{\text{bis}}$) the class of those which can be considered intrinsically as topological groups. The main results about Boolean lattices in [1] are:

(A) *In a complete Boolean lattice B, if there exists a separated topology which is compatible with the (algebraic) lattice structure and weaker than order-topology, and has a neighborhood-basis of order-closed order-convex sets, then it is determined uniquely and B is a topological group with it.*

(B) *In the above condition, B has sufficiently many continuous characters if and only if B is atomic, i.e. isomorphic to the Boolean lattice of all the subsets of a set.*

For a σ-complete Boolean lattice B and a countably additive measure μ on it, the quotient lattice of B by the sublattice $\{x;\mu(x)=0\}$ is complete and μ defines a topology as above. This is called *the complete Boolean lattice (with the intrinsic topology) associated to (B and) μ.*

In this paper, we shall consider the unitary representation of these groups.

1. *Measures and positive-definite functions.*

Proposition 1. Let B be a Boolean lattice and μ be a positive function on it, and suppose $\mu(0)=0$. Then μ is a (finitely additive) measure if and only if, for every $a\in B$. $f_a(x)=\mu(a)-2\mu(x)$ is positive-definite on the subgroup $[0,a]=\{x;\ 0\leqq x\leqq a\}$.

In fact, for a measure μ,

$$\sum_{i,j=1}^{n} f_a(x_i\triangle x_j)\alpha_i\bar{\alpha}_j=\int_a\left|\sum_{i=1}^{n}\alpha_i(1-2\varphi_{x_i})\right|^2 d\mu$$

(formal integral on the representation space, where φ_x means the characteristic function of x).

Conversely, if f_a is positive-definite, for $x\frown y=0$ and $x\smile y=a$, we have

$$\begin{vmatrix} f_a(a\triangle a), & f_a(a\triangle y), & f_a(a\triangle 0) \\ f_a(y\triangle a), & f_a(y\triangle y), & f_a(y\triangle 0) \\ f_a(0\triangle a), & f_a(0\triangle y), & f_a(0\triangle 0) \end{vmatrix}=-4\mu(a)(\mu(a)-\mu(x)-\mu(y))^2,$$

and hence $\mu(a)=\mu(x)+\mu(y)$.

This Proposition can be generalized to the relation between operator-valued measures and positive-definite operator-valued functions in Hilbert spaces.

As a special case of positive-definite operator-valued functions, *if* $V(x)$ *is a unitary representation of* B *such that* $V(1)=-1$ *and* $V(x)\geqq V(y)$ *for* $x\leqq y$, $P(x)=\dfrac{1-V(x)}{2}$ *is a projection measure.* In fact, for $x\frown y=0$ and $x\smile y=a$, $(V(x)+V(y)-V(a)-1)^2=-4(V(x)+V(y)-V(a)-1)$ and $V(x)+V(y)-V(a)-1\geqq -2$, and hence $V(x)+V(y)=V(a)+1$. But, in general, even if f is a positive-definite scalar-valued function on a finite Boolean lattice such that $f(1)=-1$, $f(0)=1$ and $f(x)\geqq f(y)$ for $x\leqq y$, $\dfrac{1-f(x)}{2}$ is not a measure.

M. A. Naimark proved that:

(I) *Every continuous positive-definite operator-valued function on a commutative locally compact group can be extended to a continuous unitary representation in an extended Hilbert space* [5].

(II) *Every countable additive operator-valued measure on a σ-complete Boolean lattice can be extended to a countable additive projection-measure in an extended Hilbert space* [6].

Recently, B. Sz.-Nagy in [7] generalized (I) by Gelfand-Raikov's method (cf. [2]), as follows:

(I′) *Every (continuous) positive-definite operator-valued function on a (topological) group can be extended to a (continuous) unitary representation in an extended Hilbert space.*

Further he led (II) in the case of Radon measures on commutative locally compact groups by use of Bochner-Stone's theorem.

Applying Proposition 1 to (I′) and noticing the fact that in the extension (I′) the order-monotonity is preserved (as we can check easily by use of the formal integral), we have:

(II′) *Every operator-valued measure on a Boolean lattice can be extended to a projection measure in an extended Hilbert space.*

Furthermore, considering the associated complete Boolean lattice, we can obtain (II), since the countable additivity means the continuity with the intrinsic topology and in the extension (I′) the continuity is preserved.

The complete Boolean lattice associated to the Lebesgue measure on the reals has no non-trivial continuous character by virtue of (B), but its measure gives a non-trivial continuous unitary representation by virtue of (I′) and Proposition 1. This provides an example that *the reduction theory* is not available.

2. *Atomicity and compacity.* Applying (B) to the theory of operator rings, we have immediately:

Proposition 2. An operator ring is isomorphic to a product of factors of type I and of finite class if and only if every commutative sublattice of its projection lattice has sufficiently many continuous (with the strong topology of operators) characters.

$P \rightarrow 1-2P$ is a topological isomorphism of the commutative projection lattice into the unitary group, and hence in the Proposition we can change the words "*every commutative sublattice of its projection lattice*" to "*every commutative subgroup of its unitary group whose elements are order* 2". A near result in the case of factors was obtained by R. V. Kadison in [4].

In particular, if the unitary group of an operator ring is compact (in this case our Proposition can be proved with a more elementary method by use of measures as in [3]), we have the reduction of that ring. This suggests a close connection between *the atomicity of compact Boolean lattices* and *the reduction of continuous unitary representations of compact groups* or *the spectral resolution of compact operators.*

References

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