2. *Fourier Series. VI. A Convergence Theorem*

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1. We have proved the following theorem [1].

Theorem 1. *If, for a fixed x*

$$\int_0^t [f(x+u)-f(x)]du=o(t) \qquad as \quad t\to 0 \tag{1}$$

and

$$\int_0^t [f(\xi+u)-f(\xi-u)]du=o\left(t/\log\frac{1}{t}\right) \qquad as \quad t\to 0 \tag{2}$$

uniformly in ξ in a neighbourhood of x, the Fourier series of $f(t)$ converges at $t=x$.

Further S. Izumi [2] has proved

Theorem 2. *If, for a fixed x*

$$\int_0^t |f(x+u)-f(x)|du=o(t) \qquad as \quad t\to 0 \tag{3}$$

and

$$n\int_0^{\pi/n} dt\left|\sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+(2k+1)\pi/n}\frac{f(v)-f(v-\pi/n)}{v}dv\right|=o(1) \tag{4}$$

as $n\to\infty$, then the Fourier series of $f(t)$ converges at $t=x$.

We shall here prove the following theorems.

Theorem 3. *If the Fourier series of $f(t)$ is summable $(C,1)$ at $t=x$ and the condition (2) holds, then the Fourier series of $f(t)$ converges at $t=x$.*

Theorem 4. *If the Fourier series of $f(t)$ is summable $(C,1)$ at $t=x$ and*

$$\int_0^t [f(\xi+u)-f(\xi-u)]du=o(t) \qquad as \quad t\to 0 \tag{5}$$

*uniformly in ξ in a neighbourhood of x, and further if**)

$$\int_0^{\pi/n}\left|\sum_{\substack{j=-[n/2]\\ j\neq 0}}^{[n/2]}\frac{\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)}{t+2j\pi/n}\right|dt=o(t) \qquad as \quad n\to\infty, \tag{6}$$

then the Fourier series of $f(t)$ converges at $t=x$.

It is known that the condition (3) implies (C, 1) summability of Fourier series of $f(t)$ at $t=x$, but the condition (1) does not so.

For the proof of these theorems we make use of the following theorem, due to W. W. Rogosinski [3]:

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*) Δ^2_h is the second difference with breadth h, and then $\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)$ $=f(x+t+(2j-1)\pi/n)-2f(x+t+2j\pi/n)+f(x+t+(2j+1)\pi/n)$.

Theorem 5. *If the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$ to $f(x)$, then*

$$\frac{1}{2}(s_n(x+\alpha_n)+s_n(x-\alpha_n))-(s_n(x)-f(x))\cos n\alpha_n\to f(x),$$

where $\alpha_n=O(1/n)$ and $s_n(x)$ is the nth partial sum of the Fourier series of $f(t)$.

We prove

Theorem 6. *If (5) holds, then*

$$\frac{1}{2}(s_n(x+2\pi/n)+s_n(x-2\pi/n))-s_n(x)\to 0.$$

This theorem is the case $\alpha_n=2\pi/n$ in Theorem 5, but the condition (5) does not imply the (C, 1) summability of the Fourier series of $f(t)$.

2. Proof of Theorem 3. Let $s_n(x)$ be the nth partial sum of the Fourier series of $f(x)$, then

$$s_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin(n+1/2)t}{2\sin t/2}dt=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin nt}{t}dt+o(1)$$

$$=\frac{1}{\pi}\sum_{k=-n}^{n}(-1)^k\int_0^{\pi/n}\frac{f(x+t+k\pi/n)}{t+k\pi/n}\sin nt\,dt+o(1)$$

$$=\frac{1}{\pi}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\left[\frac{f(x+t+2k\pi/n)}{t+2k\pi/n}-\frac{f(x+t+(2k+1)\pi/n)}{t+(2k+1)\pi/n}\right]\sin nt\,dt+o(1).$$

Let us consider

$$\delta_n(x)=\theta_n(x)-s_n(x)=\frac{1}{2}(s_n(x+\pi/2n)+s_n(x-\pi/2n))-s_n(x)$$

$$(7)\qquad =\frac{1}{2\pi}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\left[\frac{\Delta^2_{\pi/2n}f(x+t+(2k-1/2)\pi/n)}{t+2k\pi/n}-\frac{\Delta^2_{\pi/2n}f(x+t+(2k+1/2)\pi/n)}{t+(2k+1)\pi/n}\right]\sin nt\,dt+o(1).$$

Let $k\neq 0$. By the condition (2), for $0\leqq\xi<\eta\leqq\pi/n$,

$$\int_0^{\pi/n}\frac{\Delta^2_{\pi/2n}f(x+t+(2k-1/2)\pi/n)}{t+2k\pi/n}\sin nt\,dt$$

$$=\frac{n}{2k\pi}\int_\xi^\eta \Delta^2_{\pi/2n}f(x+t+(2k-1/2)\pi/n)\,dt=o\left(\frac{1}{(|k|+1)\log n}\right),$$

and similarly

$$\int_0^{\pi/n}\frac{\Delta^2_{\pi/2n}f(x+t+(2k+1/2)\pi/n)}{t+(2k+1)\pi/n}\sin nt\,dt=o\left(\frac{1}{(|k|+1)\log n}\right).$$

In the case $k=0$, using monotonity of $\sin nt/t$, we get the same estimation. Hence

$$\delta_n(x)=o\left(\sum_{k=-n}^{n}\frac{1}{(|k|+1)\log n}\right)=o(1).$$

On the other hand, by Theorem 5, $\theta_n(x)\to f(x)$, and then $s_n(x)\to f(x)$.

3. Proof of Theorem 4. Starting from (7), we get

$$\delta_n(x)=\frac{1}{2\pi}\sum_{k=-[n/2]}^{[n/2]}$$

$$\int_0^{\pi/n}\frac{\Delta^2_{\pi/2n}f(x+t+(2k-1/2)\pi/n)-\Delta^2_{\pi/2n}f(x+t+(2k+1/2)\pi/n)}{t+2k\pi/n}\sin nt\,dt$$

$$+\frac{1}{2n}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\frac{\Delta^2_{\pi/2n}f(x+t+(2k+1/2)\pi/n)}{(t+2k\pi/n)(t+(2k+1)\pi/n)}\sin nt\,dt+o(1)$$

$$=I+J+o(1).$$

By the second mean value theorem and (5), we get

$$J=\frac{n}{2}\sum_{k=-[n/2]}^{[n/2]}\frac{\theta_k}{k^2+1}\int_{\xi_k}^{\eta_k}\Delta^2_{\pi/2n}f(x+t+(2k+1/2)\pi/n)dt=o(1),$$

where $0<\theta_k\leqq 1,\ 0\leqq\xi_k<\eta_k\leqq\pi/n$.

On the other hand we put

$$I=\frac{1}{2\pi}\left[\sum_{k=1}^{[n/2]}+\sum_{k=-[n/2]}^{-1}\right]$$

$$+\frac{1}{2\pi}\int_0^{\pi/n}\frac{\Delta^2_{\pi/2n}f(x+t-\pi/2n)-\Delta^2_{\pi/2n}f(x+t+\pi/2n)}{t}\sin nt\,dt$$

$$=I_1+I_2+I_3.$$

Then by the second mean value theorem and monotonity of $\sin nt/t$

$$I_3=\frac{n}{2\pi}\int_\xi^\eta(\Delta^2_{\pi/2n}f(x+t-\pi/2n)-\Delta^2_{\pi/2n}f(x+t+\pi/2n))dt=o(1),$$

where $0\leqq\xi<\eta\leqq\pi/n$. And further

$$I_1=\frac{1}{2n}\sum_{j=1}^{[n/2]}\int_0^{\pi/n}\frac{\sin nt\,dt}{(t+2k\pi/n)(t+(2k+1)\pi/n)}$$

$$\sum_{j=1}^{k}(\Delta^2_{\pi/2n}f(x+t+(2j-1/2)\pi/n)-\Delta^2_{\pi/2n}f(x+t+(2j+1/2)\pi/n))$$

$$+\frac{1}{2\pi}\int_0^{\pi/n}\frac{\sin nt\,dt}{t+\pi}$$

$$\sum_{j=1}^{[n/2]}(\Delta^2_{\pi/2n}f(x+t+(2j-1/2)\pi/n)-\Delta^2_{\pi/2n}f(x+t+(2j+1/2)\pi/n)).$$

Since

$$\sum_{j=1}^{k}(\Delta^2_{\pi/2n}f(x+t+(2j-1/2)\pi/n)-\Delta^2_{\pi/2n}f(x+t+(2j+1/2)\pi/n))$$

$$=-[f(x+t+(2k+3/2)\pi/n)-f(x+t+(2k+1)\pi/n)]$$

$$+\sum_{j=1}^{k}[f(x+t+(2j-1)\pi/n)-2f(x+t+2j\pi/n)+f(x+t+(2j+1)\pi/n)],$$

we have

$$I_1=\frac{1}{2n}\sum_{k=1}^{[n/2]}\int_0^{\pi/n}\frac{\sin nt\,dt}{(t+2k\pi/n)(t+(2k+1)\pi/n)}\sum_{j=1}^{k}\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)$$

$$+\frac{1}{2n}\sum_{k=1}^{[n/2]}\int_0^{\pi/n}\frac{f(x+t+(2k+1)\pi/n)-f(x+t+(2k+3/2)\pi/n)}{(t+2k\pi/n)(t+(2k+1)\pi/n)}\sin nt\,dt+o(1)$$

$$=\frac{1}{2\pi}\int_0^{\pi/n}\sin nt\,dt\sum_{j=1}^{[n/2]}\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)$$
$$\sum_{k=j}^{[n/2]}\left(\frac{1}{t+2k\pi/n}-\frac{1}{t+(2k+1)\pi/n}\right)+o(1).$$

The inner sum of the last term is written as

$$\sum_{k=j}^{[n/2]}\left(\frac{1}{t+2k\pi/n}-\frac{1}{t+(2k+1)\pi/n}\right)=\frac{n}{\pi}\sum_{k=j}^{[n/2]}\left(\frac{1}{nt/\pi+2k}-\frac{1}{nt/\pi+2k+1}\right)$$
$$=\frac{n}{2\pi}\sum_{k=j}^{[n/2]}\int_{2k}^{2k+1}\frac{dx}{(nt/\pi+x)^2}=\frac{n}{4\pi}\int_{2j}^{2[n/2]+1}\frac{dx}{(nt/\pi+x)^2}+\frac{n}{4\pi}\sum_{k=j}^{[n/2]}\left(\int_{2k}^{2k+1}-\int_{2k+1}^{2k+2}\right)$$
$$=\frac{1}{2}\frac{1}{t+2j\pi/n}-\frac{1}{4(t+\pi(2[n/2]+1)/n)}$$
$$+\frac{n}{4\pi}\sum_{k=j}^{[n/2]}\int_{2k}^{2k+1}\left(\frac{1}{(nt/\pi+x)^2}-\frac{1}{(nt/\pi+x+1)^2}\right)dx$$
$$=\frac{1}{2}\frac{1}{t+2j\pi/n}-\frac{1}{4(t+\pi(2[n/2]+1)/n)}$$
$$+\frac{n}{2\pi}\sum_{k=j}^{[n/2]}\int_{2k}^{2k+1}\frac{nt/\pi+x+1/2}{(nt/\pi+x)^2(nt/\pi+x+1)^2}dx.$$

Thus we have by the second mean value theorem

$$I_1=\frac{1}{4\pi}\int_0^{\pi/n}\sum_{j=1}^{[n/2]}\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)\frac{\sin nt}{t+2j\pi/n}dt+o(1)$$
$$+\frac{n}{4\pi^2}\int_0^{\pi/n}\sum_{j=1}^{[n/2]}\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)$$
$$\left(\sum_{k=j}^{[n/2]}\int_{2k}^{2k+1}\frac{nt/\pi+x+1/2}{(nt/\pi+x)^2(nt/\pi+x+1)^2}dx\right)\sin nt\,dt$$
$$=\frac{1}{4\pi}\int_0^{\pi/n}\sum_{j=1}^{[n/2]}\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)\frac{\sin nt}{t+2j\pi/n}dt+o(1).$$

And similarly we have

$$I_2=\frac{1}{4\pi}\int_0^{\pi/n}\sum_{j=-[n/2]}^{-1}\Delta^2_{\pi/n}f(x+t+(2j-1)\pi/n)\frac{\sin nt}{t+2j\pi/n}dt+o(1).$$

Accordingly by the condition (6) we get $I_1+I_2=o(1)$. Thus we get $\delta_n(x)\to 0$ and then $s_n(x)\to f(x)$ by Theorem 5.

4. Proof of Theorem 6. Let us consider

$$\delta_n(x)=\theta_n(x)-s_n(x)=\frac{1}{2}(s_n(x+2\pi/n)+s_n(x-2\pi/n))-s_n(x)$$
$$=\frac{1}{2\pi}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\frac{\Delta^2_{2\pi/n}f(x+t+2(k-1)\pi/n)}{t+2k/n}\sin nt\,dt$$
$$-\frac{1}{2\pi}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\frac{\Delta^2_{2\pi/n}f(x+t+(2k-1)\pi/n)}{t+(2k+1)\pi/n}\sin nt\,dt+o(1)$$

$$=\frac{1}{2\pi}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\frac{\Delta^2_{2\pi/n}f(x+t+2(k-1)\pi/n)-\Delta^2_{2\pi/n}f(x+t+(2k-1)\pi/n)}{t+2k\pi/n}\sin nt\,dt$$

$$-\frac{1}{2n}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\frac{\Delta^2_{2\pi/n}f(x+t+(2k-1)\pi/n)}{(t+2k\pi/n)(t+(2k+1)\pi/n)}\sin nt\,dt+o(1)$$

$$=I-J+o(1),$$

say. By the Abel lemma

$$I=\frac{1}{2\pi}\int_0^{\pi/n}\sum_{k=-[n/2]}^{[n/2]}\frac{\pi}{n}\frac{\sin nt\,dt}{(t+2k\pi/n)(t+(2k+1)\pi/n)}$$

$$\sum_{j=-[n/2]}^{k}(\Delta^2_{2\pi/n}f(x+t+2(j-1)\pi/n)-\Delta^2_{2\pi/n}f(x+t+(2j-1)\pi/n))$$

$$+\frac{1}{2\pi}\int_0^{\pi/n}\frac{\sin nt\,dt}{t+2\pi[n/2]/2}$$

$$\sum_{j=-[n/2]}^{[n/2]}(\Delta^2_{2\pi/n}f(x+t+2(j-1)\pi/n)-\Delta^2_{2\pi/n}f(x+t+(2j-1)\pi/n))$$

$$+o(1)$$

$$=I_1+I_2+o(1),$$

where

$$\sum_{j=-[n/2]}^{k}(\Delta^2_{2\pi/n}f(x+t+2(j-1)\pi/n)-\Delta^2_{2\pi/n}f(x+t+(2j-1)\pi/n))$$

$$=[f(x+t+2(k+1)\pi/n)-f(x+t+2k\pi/n)]$$

$$-[f(x+t+(2k+3)\pi/n)-f(x+t+(2k+1)\pi/n)].$$

We can easily see that $I_2=o(1)$. Further, by the second mean value theorem,

$$I_1=\frac{1}{2n}\sum_{k=-[n/2]}^{[n/2]}\int_0^{\pi/n}\frac{f(x+t+(2k+1)\pi/n)-f(x+t+2k\pi/n)}{(t+2k\pi/n)(t+(2k+1)\pi/n)}\sin nt\,dt$$

$$=\frac{n}{2\pi^2}\sum_{k=-[n/2]}^{[n/2]}\frac{\theta_k}{(2k+1/2)(2k+1)}$$

$$\int_{\xi_k}^{\eta_k}[f(x+t+(2k+1)\pi/n)-f(x+t+2k\pi/n)]\,dt$$

where $0<\theta_k\leqq 1$, $0\leqq\xi_k<\eta_k\leqq\pi/n$, and hence

$$I_1=o(1)\sum_{k=-[n/2]}^{[n/2]}\frac{1}{k^2+1}=o(1).$$

Thus we have $I=o(1)$.

Similarly

$$J=\frac{n}{2\pi^2}\sum_{k=-[n/2]}^{[n/2]}\frac{\theta'_k}{(2k+1/2)(2k+1)}\int_{\xi'_k}^{\eta'_k}\Delta^2_{2\pi/n}f(x+t+(2k-1)\pi/n)\,dt$$

$$=o(1).$$

Accordingly we get $\delta_n(x)=o(1)$ in the interval $(0,\ \pi/2)$, and then in the interval $(0,\ 2\pi)$.

After W. W. Rogosinski [3], $\theta_n(x) \to f(x)$, a.e., and hence $s_n(x) \to f(x)$, a.e.

References

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