23. *Divergent Integrals as Viewed from the Theory of Functional Analysis. I*

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§ 1. *Introduction.* Let the complex valued function $f(z,\lambda)$ be defined for a real number λ, $(a\leq\lambda\leq b)$ and for a complex number z on a domain D and also on another domain D_1. We assume that the integral $\int_a^b f(\lambda,z)d\lambda$ converges for $z\in D$ and diverges for $z\in D_1$. We denote the convergent integral for $z\in D$ by $f(z)$.

Now in many branches of analysis, it is often necessary or convenient to use a function $f^*(z)$ on $z\in D_1$ which in some senses corresponds to $f(z)$ on $z\in D$. For example, (i) when $f(z)$ is analytic on D, the analytic extension in D_1 of $f(z)$ may be taken as $f^*(z)$, (ii) when $f(z,x)$ is a solution for $z\in D$ of some differential or integral equations which contain z as a parameter, the solution for $z\in D_1$ may be taken as $f^*(z)$. To derive the functions $f^*(z)$ in such cases, we have many classical methods: changings of the contour of integral, or various methods of summation, or other limiting processes.

In this and the following papers we always view such divergent integrals from the theory of functional analysis. We construct a functional space Φ whose elements for example are the functions $\varphi(\sigma,\tau)$ defined on D_1 or on both D and D_1 having suitable properties there (here σ and τ are respectively the real part and the imaginary part of z).

Now the mapping $\lambda\to f(\lambda,z)$ defines a mapping from $a\leq\lambda\leq b$ to Φ', where Φ' is a dual space of Φ. We regard the divergent integral $\int_a^b f(\lambda,z)d\lambda$ for $z\in D_1$ as the integral in Φ'. If it converges weakly or strongly we can examine whether the functional $f^*(z)$ defines a function or not and we can investigate its properties also.

We consider particularly about the following type of divergent integrals (1), though, of course, the similar method will be able to be adopted for other sorts of kernels, dimensions of λ or intervals of the integration by selecting suitable functional spaces Φ.

This type of integrals relates to some of the most important parts of classical analysis, i.e. the Laplace transform, the power series development, the analytic continuation, some sorts of approximations, and some differential or integral equations (see § 2).

In this paper we show some sufficient conditions for these kernels, which will be used in the following papers. We also give some examples from the Laplace transform in connection with the problem of analytic continuation. However the detailed discussion of many other applications will be done in the following papers.

We consider the following divergent integrals

$$v^*(k)=\int_0^\infty \exp(iks)v(\sigma,\tau,s)\,ds \tag{1}$$

where $k=\sigma+i\tau$. We denote the domain $-\infty<\sigma<+\infty$, $\tau_1\leq\tau\leq\tau_2$ in the plane $R_\sigma\times R_\tau$ by D_1. $v(\sigma,\tau,s)$ is defined on D_1 and satisfies the condition (v) or $(v)'$ (see §§ 4, 5).

§ 2. *Relations to other problems.*

(1) Relation to the Laplace transform. When v is dependent only on s, our integral is a Laplace integral. In this case, the condition (v) or $(v)'$ is not necessary. Indeed, as remarked in § 5, even if $v(s)$ is not a function we can apply our method for an arbitrary distribution and in any strip D_1 on $R_\sigma\times R_\tau$.

(2) Relations to Borel's integral. Let $f(z)=\sum_{n=0}^\infty a_n z^n$ be analytic when $|z|\leq r$. Then $\phi(z)=\sum_{n=0}^\infty a_n z^n/n!$ is an integral function. We put $f_1(z)=\int_0^\infty e^{-t}\phi(zt)dt$ and it holds $f_1(z)=f(z)$ when $|z|<r$. $f_1(z)$ represents an analytic function in a more extended region (i.e. in the Borel's polygon) than the circle $|z|<r$.

We put $\phi(zt)=e^{zt}v(zt)$. If $v(zt)$ satisfies (v) or (v), our methods are valid outside the polygon. So in this case we have more detailed knowledge about the analytic extension of the power series than obtained by Borel's method.

(3) We consider the following singular integral equations of Volterra's type containing a parameter k.

$$u(k,r)=u_0(k,r)+\int_r^\infty \exp\{ik(s-r)\}\,V(k,r,s)u(k,s)ds \tag{2}$$

Formal solution is expressed using the resolvent Γ

$$u(k,r)=u_0(k,r)+\int_r^\infty \Gamma(k,r,s)u_0(k,s)ds \tag{3}$$

where

$$\Gamma(k,r,s)=\exp\{ik(s-r)\}\,w(k,r,s).$$

$w(k,r,s)$ is determined by $V(k,r,s)$ and if $w(k,r,s)u_0(k,s)$ satisfies our condition (v) or $(v)'$, we can define the functional solution for $k\in D_1$ where the formal solution (3) diverges in a proper sense. We can utilize this functional solution (a) to obtain the proper solution, (b) to examine an analytic continuation in the domain D_1 from the domain D where a proper solution $u(k,r)$ is regular, (c) to examine a solution for $k\in D_1$ of the differential equation $L[k,u]=0$ which is equivalent to the integral equation (2) for $k\in D$.

(4) Other related problems.

(a) To investigate a solution of differential equation outside the circle of convergence when we solve it by the power-series expansions, or by other approximations.

(b) To investigate directly properties of a function which have been examined (sometimes step by step) by many sorts of methods of summation.

(c) To extend the method of solving differential or integral equations using Laplace transformation, or to extend other applications of Laplace transform.

(5) Remark. We mentioned hitherto the way in which we regard $f(\lambda, z)$ as a functional ($\in \Phi'(\sigma, \tau)$) having a parameter λ, but it is also possible in some cases to regard $f(\lambda, z)$ as a functional ($\in \Phi'(\lambda)$) having a parameter z (see § 7). This is often available when a divergence of integral causes from a singularity of $f(\lambda, z)$ at a finite point $\lambda_0 (a \leq \lambda_0 \leq b)$.

§ 3. *The space* Φ. We take the functional space $\mathfrak{D}_K(\tau)$ defined by L. Schwartz [1] whose elements have carriers in the compact set $K=[\tau_1, \tau_2]$, and denote it simply by $\mathfrak{D}(\tau)$. We take on the other hand the functional space $Z(\sigma)$ used by Gelfand-Silov [2] or Ehrenpreis [3]. We take the biprojective tensor product [4] $\mathfrak{D}(\tau)\hat{\hat{\otimes}} Z(\sigma)$ of these two spaces, and denote it by Φ. The space Φ has the following properties.

1° $\Phi = Z(\sigma)\hat{\otimes}\mathfrak{D}(\tau)$, i.e. Φ is equal to the projective tensor product [4].

2° Let $l>0$; by K_l we denote the closed interval in R_σ, center at the origin, and length $2l$. By $\mathfrak{D}_l$ we denote the space of infinitely differentiable functions on $R(\sigma)$ which vanish outside K. By $d\sigma$, we denote the usual measure on R, divided by $\sqrt{(2\pi)}$. For any function f in $\mathfrak{D}_l$ we define its Fourier transform $F=\mathfrak{F}(f)$ by $F(z)=\int f(\sigma)e^{-i\sigma z}d\sigma$. It is known that F is an entire function of exponential type $\leq l$ which is rapidly decreasing on R. If we denote the functional space of F by $\boldsymbol{D}_l(\sigma)$, then we can see that $Z(\sigma)$ is the inductive limit of $\boldsymbol{D}_l(\sigma)$: $Z(\sigma)=\bigcup_l \boldsymbol{D}_l(\sigma)$ [3].

3° The space $\mathfrak{D}(\sigma)$ is dense in the space $\mathcal{S}(\sigma)$ in the topology of $\mathcal{S}$. ($\mathcal{S}$ means the space defined by L. Schwartz [1].) Hence the space $Z(\sigma)$ is also dense in the space $\mathcal{S}(\sigma)$ in the topology of $\mathcal{S}$.

4° Both the spaces $\mathfrak{D}(\sigma)$ and $Z(\sigma)$ are reflexive.

5° For any element $d' \in \mathfrak{D}'$ and an element $\varphi \in \Phi$, we can naturally make correspond an element $\langle d', \varphi\rangle$ of Z such that $\langle d', \varphi\rangle = \sum_{j=1}^\infty \langle d', \varphi_j(\tau)\rangle \varphi_j(\sigma)$ for $\varphi = \sum_{j=1}^\infty \varphi_j(\sigma)\varphi_j(\tau)$. The element $\langle d', \varphi\rangle$ is uniquely determined by φ independently of the expression of $\varphi : \varphi = \sum_j \varphi_j(\sigma)\varphi_j(\tau)$. Moreover the mapping $\varphi \to \langle d', \varphi\rangle$ from $\Phi(\sigma, \tau)$ to $Z(\sigma)$ is continuous.

Similarly for any element $z'\in Z'$, and for any element $\varphi\in\Phi$, it holds $\langle z',\varphi\rangle\in\mathfrak{D}(\tau)$, and the mapping $\varphi\to\langle z',\varphi\rangle$ from $\Phi(\sigma,\tau)$ to $\mathfrak{D}(\tau)$ is continuous.

6° We take specially Dirac's measure at τ^0, δ_{τ^0} as d' in the property 5°. Then we see $\varphi(\sigma,\tau^0)\in Z(\sigma)$.

Similarly taking Dirac's measure at σ^0 as z' we see $\varphi(\sigma^0,\tau)\in\mathfrak{D}(\tau)$. Thus we see that any element φ of Φ is a function $\varphi(\sigma,\tau)$ such that for a fixed τ, $\varphi(\sigma,\tau)$ belongs to $Z(\sigma)$ and for a fixed σ, $\varphi(\sigma,\tau)$ belongs to $Z(\sigma)$.

7° Hitherto we have denoted simply by $\mathfrak{D}(\tau)$ the space $\mathfrak{D}_K(\tau)$ where $K=(\tau_1,\tau_2)$. For a moment, we use the notation $\mathfrak{D}_l(\tau)$ and denote $\bigcup_l\mathfrak{D}_l(\tau)$ by $\mathfrak{D}(\tau)$ as usual. Now we can see the following inclusion relation.

$$Z(\sigma)\otimes\mathfrak{D}(\tau)\subset Z(\sigma)\widehat{\widehat{\otimes}}\mathfrak{D}(\tau)\subset\mathcal{S}(\sigma,\tau),$$

where $\mathcal{S}$ (σ,τ) means the space $\mathcal{S}$ given by L. Schwartz on the domain $R_\sigma\times R_\tau$. We can see this relation using the property $Z(\sigma)\widehat{\widehat{\otimes}}\mathfrak{D}(\tau)=Z(\sigma)\widehat{\otimes}\mathfrak{D}(\tau)$. On the other hand $Z(\sigma)\otimes\mathfrak{D}(\tau)$ is dense (in $\mathcal{S}$ topology) in $\mathcal{S}(\sigma)\otimes\mathcal{S}(\tau)$ hence is dense in $\mathcal{S}(\sigma,\tau)$. So we can see $Z(\sigma)\widehat{\widehat{\otimes}}\mathfrak{D}(\tau)$ is dense in $\mathcal{S}$ topology in $\mathcal{S}(\sigma,\tau)$. So for any $\varphi\in\mathfrak{D}(\sigma,\tau)$ there exists a sequence $\{\varphi_j(\sigma,\tau)\}$ such that $\varphi_j\in Z(\sigma)\widehat{\widehat{\otimes}}\mathfrak{D}(\tau)$ and φ_j converges to φ in the topology $\mathcal{S}$.

Especially for a φ which belongs to $\mathfrak{D}_L(\sigma,\tau)$ where L means a compact set $(\tau_1\leq\tau\leq\tau_2,\sigma_1\leq\sigma\leq\sigma_2)$ in $R_\sigma\times R_\tau$, we can select such a sequence $\{\varphi_j(\sigma,\tau)\}$ particularly from the space Φ.

§ 4. *The condition* (v) *and the convergence.*

The condition (v) is the following:

(1) $v(\sigma,\tau,s)$ is a continuous function of 3 variables σ,τ,s, on $(\sigma,\tau)\in D_1$, $0\leq s<\infty$.

(2) For any $\varphi\in\Phi$, $v\varphi$ belongs to the space Φ.

(3) For any fixed $\varphi(\in\Phi)$ and fixed τ, after § 3 property 6°, $\varphi(\sigma,\tau)\in\boldsymbol{D}_l(\sigma)$. So if v satisfies above condition (2), there exists a positive number l' such that $v\varphi\in\boldsymbol{D}_{l'}(\sigma)$. Now we demand that

(3.1) there exists *sup* l' for $0\leq s$, and

(3.2) for any fixed φ and τ and any s_0, there exists a positive number M such that $|v(\sigma,\tau,s)\varphi(\sigma,\tau)|<M$ for $0\leq s<s_0$. (M is dependent on s_0 and is independent of σ.)

Theorem 1. *If the function* v *in the integral* (1) *satisfies the condition* (v), *then for any* $\varphi\in\Phi$ *the integral* (1) *is convergent.*

Proof. We put

$$v_1(\sigma,s)=\int_{\tau_1}^{\tau_2}\exp(-\tau s)v(\sigma,\tau,s)\varphi(\sigma,\tau)d\tau,$$

and

$$v_2(s,t)=\int_{-\infty}^{+\infty}\exp(i\sigma s)v_1(\sigma,t)d\sigma.$$

Using the mean value theorem of integrations we can see that $v_1(\sigma,s)\in \boldsymbol{D}_l(\sigma)$ where l is common for all s. So it follows that $v_2(s,t)\in\mathfrak{D}_l(s)$ where l can be chosen common for all s.

We can see also using Arzelà's theorem that $v_2(s,t)$ is a continuous function of 2 variables (s,t). So the integral

$$\langle v^*,\varphi\rangle=\int_0^{\infty}v_2(s,s)ds=\int_0^{l}v_2(s,s)ds \quad \text{exists.}$$

§ 5. *Continuity*. The condition $(v)'$ for the continuity of v^* on Φ:

(1) The same as the first one of the conditions (v).

(2) v is a multiplicator of Φ, i.e. for any $\varphi\in\Phi$, $v\varphi$ belongs to Φ and moreover the mapping from Φ to Φ, $\varphi\rightarrow v\varphi$ is continuous.

(3) The continuous mapping $\varphi\rightarrow v\varphi$ is uniform in any finite interval $0\leq s<s_0$.

Theorem 2. *If v in the integration (1) satisfies the condition $(v)'$, v^* is a continuous linear functional on Φ.*

Proof. 1° We can see that if a sequence $\{\varphi_j(\sigma,\tau)\}$ of Φ converges to 0 in the space Φ, $\varphi_j(\sigma,\tau)$ converges to 0 for a fixed τ in the space $Z(\sigma)$ uniformly in the closed interval $[\tau_1,\tau_2]$. This causes from the fact that the set of Dirac's measure δ_{τ^0} at τ^0, $(\tau_1\leq\tau^0\leq\tau_2)$ is an equicontinuous set in $\mathfrak{D}'$.

So it follows that there exists a positive number l such that $\varphi_j(\sigma,\tau)\in\boldsymbol{D}_l(\sigma)$ for any integer j and any $\tau(\tau_1\leq\tau\leq\tau_2)$. (The same holds true for converging filter $\{\varphi_a\}$.)

2° Since the set $\{\exp(i\sigma s)\,|\,0\leq s\}$ is an equicontinuous set in $Z'(\sigma)$ and the set $\{\exp(-\tau t)\,|\,0\leq t\leq l\}$ is an equicontinuous set in $\mathfrak{D}'(\tau)$ $(\tau_1\leq\tau\leq\tau_2)$ and v satisfies $(v)'$ (3), we can see the following:

Putting $v_{2j}(s,t,r)=\langle\exp(i\sigma s)\otimes\exp(-\tau t),\ v(\sigma,\tau,r)\times\varphi_j(\sigma,\tau)\rangle$, $v_{2j}(s,t,r)$ converges to 0 uniformly for s $(0\leq s)$, t $(0\leq t\leq l)$, and r $(0\leq t\leq l)$.

3° From 1° and 2° we can easily see

$$lim_j\langle v^*,\varphi_j\rangle=lim_j\int_0^{\infty}v_{2j}(s,s)ds=lim_j\int_0^{l}v_{2j}(s,s)ds=0.$$

Examining the above proofs we can see easily the truth of the following theorem:

Theorem 3. *In the case $v(\sigma,\tau,s)=\sum_{j=1}^{n}u_j(\sigma,\tau)w_j(s)$, the conditions (v) and $(v)'$ can be weakened as follows:*

$(\bar{v})$ (1) *$w_j(s)$ is locally summable.*

(2) *For any $\varphi\in\Phi$ it follows that $u_j\varphi\in\Phi$.*

$(\bar{v})'$ (1) *$w_j(s)$ is locally summable.*

(2) *u_j is a multiplicator of Φ.*

Corollary. *If $v(s)$ is locally summable, its Laplace transform $f(z)=\int_0^\infty e^{-sz}v(s)ds$ (in our sense) exists on the whole plane.*

Remark. We have hitherto considered only about the function $v(\sigma, \tau, s)$, but our treatment can be easily extended to the case when $v=\sum_j u_j(\sigma, \tau)w_j$ where w_j is a distribution on $\mathfrak{D}(s)$. The detailed discussion about this extension, however, will be done in the following paper.

Particularly in the case v is a distribution on $\mathfrak{D}(s)$, v^* represents its Laplace transform. So we can define Laplace transform of an arbitrary distribution on the whole plane.

References

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