78. *Inferior Limit of a Sequence of Potentials*

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1. In a locally compact space Ω we consider a sequence of potentials of positive measures. In case that Ω is the τ-dimensional euclidean space R^{τ} ($\tau\geqq 2$), a fundamental theorem, which was proved by Brelot [1], asserts that the inferior limit of a sequence of newtonian potentials is equal to a potential of a positive measure in the complement of a exceptional set E of inner capacity zero. Cartan [4], using the energy principle, showed that the set E is of outer capacity zero. Recently Brelot has proved that this fact follows from Choquet's result [5] on the capacitability of Borel sets. The problem of capacitability in the potential theory in a locally compact space has not yet been solved, and so in this note we shall prove under an additional condition that E is of outer capacity zero (see Brelot and Choquet [3]).

2. Let Ω be a locally compact space. We consider always positive measures μ in Ω with compact carrier, denoted by S_μ. Let $\Phi(P, Q)$ be a positive, symmetric, continuous and real valued function defined on $\Omega\times\Omega$, which is finite except at the points of the diagonal set of $\Omega\times\Omega$. The potential of μ is defined by

$$U^{\mu}(P)=\int\Phi(P, Q)\,d\mu(Q).$$

In this paper μ will be called *admissible* on a compact set K, if $S_\mu\subset K$ and $U^{\mu}(P)\leqq 1$ everywhere in Ω. The supremum of the total masses of admissible measures on K is defined to be the capacity of K and denoted by $\mathrm{cap}\,(K)$. The inner capacity $\mathrm{cap}_i\,(A)$ of $A\subset\Omega$ is equal to $\sup\mathrm{cap}\,(K)$ for compact $K\subset A$ and the outer capacity $\mathrm{cap}_e\,(A)$ is equal to $\inf\mathrm{cap}_i\,(\delta)$ for open $\delta\supset A$. Hence, for every open set δ, we have $\mathrm{cap}_i\,(\delta)=\mathrm{cap}_e\,(\delta)$. We shall designate the common value of these two capacities by $\mathrm{cap}\,(\delta)$. We say that a property holds quasi everywhere in Ω if it holds at each point of Ω except at the points of a set of outer capacity zero.

Definition 1. We shall say that a potential U^{μ} is *quasi continuous* in Ω, if, for any positive number ε, there is an open set δ_ε such that $\mathrm{cap}\,(\delta_\varepsilon)\leqq\varepsilon$ and the restriction of U^{μ} to $\Omega-\delta_\varepsilon$ is continuous.

Definition 2. We say that Φ satisfies the *quasi continuity principle*, if the continuity of the restriction of any potential U^{μ} to S_μ implies the quasi continuity of U^{μ} in Ω.

Clearly the quasi continuity principle follows from the continuity principle. (For the continuity principle, see Ohtsuka [9–11], Kishi [7], Choquet [6], Ninomiya [8].)

At first we shall assume the quasi continuity principle and prove the following

Theorem 1. *Let μ_n $(n=1,2,\cdots)$ be measures on a compact set K such that $U^{\mu_n}\leqq M<+\infty$ in Ω. If $\{\mu_n\}$ converges vaguely to μ, then we have*

$$\varliminf U^{\mu_n} = U^{\mu}$$

quasi everywhere in Ω.

3. We shall prove two lemmas for later use.

Lemma 1 (Brelot [2, Lemma 5]). *Let μ_n $(n=1,2,\cdots)$ be measures on a compact set K such that $U^{\mu_n}\leqq M<+\infty$. If $\{\mu_n\}$ converges vaguely to μ and a potential U^{ν} is quasi continuous in Ω and $U^{\nu}\leqq 1$, then we have*

$$\lim \int U^{\mu_n}\,d\nu=\int U^{\mu}\,d\nu.$$

Proof. Obviously $\int U^{\mu}\,d\nu\leqq\varliminf\int U^{\mu_n}d\nu$. Hence it is sufficient to show $\varlimsup\int U^{\mu_n}d\nu\leqq\int U^{\mu}d\nu$. U^{ν} being quasi continuous, for any $\varepsilon>0$, we can find an open set δ_ε such that $\mathrm{cap}\,(\delta_\varepsilon)\leqq\varepsilon$ and the restriction of U^{ν} to $\Omega-\delta_\varepsilon$ is continuous. Put

$$f=\begin{cases} U^{\nu} & \text{on} \quad \Omega-\delta_\varepsilon \\ 0 & \text{in} \quad \delta_\varepsilon. \end{cases}$$

Then f is upper semi-continuous. Hence we have a continuous function g such that $g\geqq f$ and $\int g\,d\mu\leqq\int f\,d\mu+\varepsilon=\int_{\Omega-\delta_\varepsilon} U^{\nu}\,d\mu+\varepsilon$. We can see that

$$\varlimsup\int_{\Omega-\delta_\varepsilon} U^{\nu}\,d\mu_n=\varlimsup\int f\,d\mu_n\leqq\lim\int g\,d\mu_n=\int g\,d\mu$$
$$\leqq\int_{\Omega-\delta_\varepsilon} U^{\nu}\,d\mu+\varepsilon\leqq\int U^{\nu}\,d\mu+\varepsilon.$$

On the other hand it is easily seen that

$$\mu_n(\delta_\varepsilon)\leqq M\varepsilon \quad \text{and} \quad \int_{\delta_\varepsilon} U^{\nu}\,d\mu_n\leqq M\varepsilon.$$

Therefore

$$\varlimsup\int U^{\nu}\,d\mu_n\leqq\int U^{\nu}\,d\mu+(M+1)\varepsilon.$$

Consequently we have

$$\varlimsup\int U^{\mu_n}d\nu=\varlimsup\int U^{\nu}\,d\mu_n\leqq\int U^{\nu}\,d\mu=\int U^{\mu}\,d\nu.$$

Lemma 2 (Cartan [4, Proposition 5]). *Every potential U^{μ} is* *quasi continuous in Ω.*

Proof. Since the set of points P such that $U^{\mu}(P)=+\infty$ is a G_{δ} set of outer capacity zero, we may suppose that U^{μ} is finite in Ω. For any $\varepsilon>0$ and for any positive integer n, by Lusin's theorem, there is a compact set K_n such that $\mu(\Omega-K_n)<\frac{\varepsilon}{2\cdot 4^n}$ and U^{μ} is continuous on K_n. Then the potential U^{μ_n} of the restriction μ_n of μ to K_n is continuous on K_n, and hence, by our quasi continuity principle, U^{μ_n} is quasi continuous in Ω. Therefore, we have an open set δ_n such that the restriction of U^{μ_n} to $\Omega-\delta_n$ is continuous and $\operatorname{cap}(\delta_n) \leqq \frac{\varepsilon}{2^{n+1}}$. Put

$$B_n=\left\{P\in\Omega-\delta_n;\ U^{\mu}(P)-U^{\mu_n}(P)>\frac{1}{2^n}\right\}$$

Then B_n is open in $\Omega-\delta_n$ and $B_n\smile\delta_n$ is open in Ω. Hence

$$\operatorname{cap}(B_n\smile\delta_n)\leqq\operatorname{cap}_i(B_n)+\operatorname{cap}(\delta_n)\leqq\operatorname{cap}_i(B_n)+\frac{\varepsilon}{2^{n+1}}$$

We shall show that $\operatorname{cap}_i(B_n)\leqq\frac{\varepsilon}{2^{n+1}}$. For any compact subset e of B_n, let γ be admissible on e. Then

$$\frac{1}{2^n}\gamma(e)\leqq\int(U^{\mu}-U^{\mu_n})\,d\gamma=\int_{\Omega-K_n}U^{\gamma}\,d\mu$$
$$\leqq\mu(\Omega-K_n)<\frac{\varepsilon}{2\cdot 4^n},$$

whence

$$\gamma(e)<\frac{\varepsilon}{2^{n+1}}\quad\text{and}\quad\operatorname{cap}(e)\leqq\frac{\varepsilon}{2^{n+1}}$$

Thus we have seen that $\operatorname{cap}_i(B_n)\leqq\frac{\varepsilon}{2^{n+1}}$ and hence $\operatorname{cap}(B_n\smile\delta_n)\leqq\frac{\varepsilon}{2^n}$, and that $\operatorname{cap}(\delta_\varepsilon)\leqq\varepsilon$, where $\delta_n=\bigcup(B_n\smile\delta_n)$. Then it follows that the restriction of U^{μ} to $\Omega-\delta_\varepsilon$ is continuous, because, on $\Omega-\delta_\varepsilon$, $0\leqq U^{\mu}-U^{\mu_n}\leqq\frac{1}{2^n}$ and U^{μ_n} is continuous.

By Lemmas 1 and 2 we have

Corollary. *If μ_n $(n=1,2,\cdots)$ are positive measures on a compact set such that $U^{\mu_n}\leqq M<+\infty$ and that $\{\mu_n\}$ converges vaguely to μ and if a potential $U^{\nu}\leqq 1$, then we have*

$$\lim\int U^{\mu_n}\,d\nu=\int U^{\mu}\,d\nu.$$

4. Proof of Theorem 1. As $\{\mu_n\}$ converges vaguely to μ, $U^{\mu}\leqq\underline{\lim}\,U^{\mu_n}$ everywhere in Ω. Hence it is sufficient to prove that $U^{\mu}\geqq\underline{\lim}\,U^{\mu_n}$ quasi everywhere in Ω. Put $V_n=\inf(U^{\mu_n},U^{\mu_{n+1}},\cdots)$ and $V_{n,m}=\min(U^{\mu_n},\cdots,U^{\mu_m})$ $(m\geqq n)$. Then the sequence $V_{n,m}$ $(m=n,$

$n+1,\cdots$) decreases to V_n as $m\to\infty$ and the sequence V_n increases to $V=\underline{\lim}\, U^{\mu_n}$ as $n\to\infty$. For any $\varepsilon'>0$ we have an open set $\delta_{\varepsilon'}$ such that $\text{cap}(\delta_{\varepsilon'})\leqq\varepsilon'$ and each U^{μ_n} and U^{μ} are continuous on $\Omega-\delta_{\varepsilon'}$ by Lemma 2. For any positive number ε, we put

$$E_{n,m}(\varepsilon)=\{P;\ V_{n,m}(P)-U^{\mu}(P)>\varepsilon\}$$

and

$$E^{\varepsilon'}_{n,m}(\varepsilon)=\{P\in\Omega-\delta_{\varepsilon'};\ V_{n,m}(P)-U^{\mu}(P)>\varepsilon\}.$$

$E^{\varepsilon'}_{n,m}(\varepsilon)$ is open in $\Omega-\delta_{\varepsilon'}$ and $E^{\varepsilon'}_{n,m}(\varepsilon)\smile\delta_{\varepsilon'}$ is open in Ω. Hence

$$\begin{aligned}\text{cap}_e(E_{n,m}(\varepsilon))&\leqq\text{cap}(E^{\varepsilon'}_{n,m}(\varepsilon)\smile\delta_{\varepsilon'})\leqq\text{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))+\text{cap}(\delta_{\varepsilon'})\\&\leqq\text{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))+\varepsilon'.\end{aligned}\tag{1}$$

We shall prove that $\lim_m\text{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))=0$. We can see immediately that $E^{\varepsilon'}_{n,m+1}(\varepsilon)\subset E^{\varepsilon'}_{n,m}(\varepsilon)$ and $\overline{E^{\varepsilon'}_{n,m+1}(\varepsilon)}\subset E^{\varepsilon'}_{n,m}\left(\frac{\varepsilon}{2}\right)$. In fact, if $P^{(k)}\in E^{\varepsilon'}_{n,m+1}(\varepsilon)$ and $P^{(k)}$ tends to P_0, then it follows that $P_0\in\Omega-\delta_{\varepsilon'}$ and that $\lim_k V_{n,m+1}(P^{(k)})=V_{n,m+1}(P_0)$ and $\lim_k U^{\mu}(P^{(k)})=U^{\mu}(P_0)$. If $\lim_m\text{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))=\alpha>0$, we should have, for any $m\geqq n$, and admissible measure $\gamma_{n,m}$ on a compact subset $e_{n,m}$ of $E^{\varepsilon'}_{n,m}(\varepsilon)$ such that $\gamma_{n,m}(e_{n,m})\geqq\frac{\alpha}{2}$. As $\text{cap}_i(E^{\varepsilon'}_{n,n}(\varepsilon))\geqq\text{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))\geqq\gamma_{n,m}(e_{n,m})$, a subsequence $\{\gamma_{n,m'}\}$ of $\{\gamma_{n,m}\}$ converges vaguely to a positive measure γ, whose total mass is obviously not smaller than $\frac{\alpha}{2}$. S_γ is contained in $E^{\varepsilon'}_{n,m}\left(\frac{\varepsilon}{2}\right)$ for every sufficiently large m; otherwise, there would be $P_0\in S_\gamma-E^{\varepsilon'}_{n,m_0}\left(\frac{\varepsilon}{2}\right)$ for some m_0, hence there would be a neighborhood δ of P_0 such that $\delta\frown E^{\varepsilon'}_{n,m_0+1}(\varepsilon)=\phi$. Then $\gamma(\delta)>0$ and $\gamma_{n,m'}(\delta)=0$ for every $m'\geqq m_0+1$, which is absurd. Since $S_\gamma\subset E^{\varepsilon'}_{n,m}\left(\frac{\varepsilon}{2}\right)$, we have

$$\frac{\alpha\varepsilon}{4}\leqq\frac{\varepsilon}{2}\gamma(\Omega)\leqq\int(V_{n,m}-U^{\mu})\,d\gamma\leqq\int(U^{\mu_m}-U^{\mu})\,d\gamma\tag{2}$$

for every sufficiently large m. On the other hand we have $\lim\int U^{\mu_m}d\gamma=\int U^{\mu}d\gamma$ by Corollary. This contradicts (2). Consequently, $\lim_m\text{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))=0$. Therefore, from (1), we see that $\lim_m\text{cap}_e(E_{n,m}(\varepsilon))\leqq\varepsilon'$, and hence $\lim_m\text{cap}_e(E_{n,m}(\varepsilon))=0$.

Now we put

$$E_n(\varepsilon)=\{P;\ V_n(P)-U^{\mu}(P)>\varepsilon\}$$

and

$$E(\varepsilon)=\{P;\ V(P)-U^{\mu}(P)>\varepsilon\}.$$

Then, as $E_n(\varepsilon)\subset E_{n,m}(\varepsilon)$ and $E(\varepsilon)=\bigcup E_n(\varepsilon)$, we have $\text{cap}_e(E_n(\varepsilon))=0$ and $\text{cap}_e(E(\varepsilon))=0$. Hence, by the usual argument, we see that $U^{\mu}(P)\geqq V(P)$ quasi everywhere in Ω.

5. In Theorem 1, we have required the uniform boundedness of U^{μ_n} $(n=1,2\cdots)$; when Φ satisfies the continuity principle, this condition, the uniform boundedness of U^{μ_n} $(n=1,2,\cdots)$, is not necessary.

Theorem 2. *Let Φ be a kernel function which satisfies the continuity principle. Let μ_n $(n=1,2,\cdots)$ be measures on a compact set K. If $\{\mu_n\}$ converges vaguely to μ, then we have*

$$\underline{\lim}\, U^{\mu_n} = U^{\mu}$$

quasi everywhere in Ω.

To prove this theorem, we shall prove, at first, the following

Lemma 3. *Let μ_n be measures on a compact set. If $\{\mu_n\}$ converges vaguely to μ and a potential $U^{\gamma}\leqq 1$, then it holds that $\int U^{\mu}d\gamma = \int V d\gamma$, where $V=\underline{\lim}\, U^{\mu_n}$.*

Proof. We assert that $\gamma(E)=0$, where

$$E=\{P;\ V(P)-U^{\mu}(P)>0\}.$$

In fact, if $\gamma(E)=\alpha>0$, we can take a compact set $e\subset E$ such that $\gamma(e)\geqq\frac{\alpha}{2}$. Then, as the restriction γ' of the measure γ to e is admissible on e, we have $\mathrm{cap}_i(E)\geqq \mathrm{cap}(e)\geqq\frac{\alpha}{2}$. This contradicts the fact that E is of inner capacity zero (see Ohtsuka [12], Brelot and Choquet [3]). Hence we get $\int U^{\mu}d\gamma=\int_{\Omega-E}U^{\mu}d\gamma=\int_{\Omega-E}V d\gamma=\int V d\gamma$.

Now we shall prove Theorem 2. We proceed in the same way as in the proof of Theorem 1. If $\lim_m \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))=\alpha>0$, then we have an admissible measure γ, for which the inequality

$$\frac{\alpha\varepsilon}{4}\leqq\int(V_{n,m}-U^{\mu})\,d\gamma$$

holds for every sufficiently large m. Here we let m tend to infinity, and we have

$$\frac{\alpha\varepsilon}{4}\leqq\int(V_n-U^{\mu})\,d\gamma\leqq\int(V-U^{\mu})\,d\gamma. \tag{3}$$

The last integral of (3) is equal to zero by Lemma 3, which is impossible. Consequently, we have $\lim_m \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon))=0$. Then, we can prove, analogously as in the proof of Theorem 1, that $V=U^{\mu}$ quasi everywhere in Ω.

6. Now we consider a family of potentials $\{U^{\mu_i}\}$ $(i\in I)$ and its lower envelope. Using Brelot and Choquet's method [3] we can prove

Theorem 3. *Let Ω be a locally compact space which has a countable base of open sets and Φ be a kernel function which satisfies the continuity principle. Let $\{\mu_i\}$ $(i\in I)$ be a family of positive measures. Suppose that*

a) S_{μ_i} $(i \in I)$ *is contained in a compact set* K;
b) *the total mass* $\mu_i(K) \leqq M$ *for each* $i \in I$;
c) *for any two potentials* U^{μ_1} *and* U^{μ_2} *of the family* $\{U^{\mu_i}\}$ $(i \in I)$, *there exists a potential* U^{μ_3} *in this family such that* $U^{\mu_3} \leqq \min(U^{\mu_1}, U^{\mu_2})$.

Then we can find a positive measure μ *such that*

$$U^{\mu} = \inf_{i \in I} U^{\mu_i}$$

quasi everywhere in Ω.

Proof. As Brelot and Choquet have shown, we can take a subsequence $\{\mu_n\}$ from $\{\mu_i\}$ $(i \in I)$ such that $\{U^{\mu_n}\}$ is decreasing and $\{\mu_n\}$ converges vaguely to a positive measure μ and that

$$U^{\mu} \leqq \inf_{i \in I} U^{\mu_i} \leqq \lim_{n} U^{\mu_n}.$$

Then, by Theorem 2, $U^{\mu} = \lim_{n} U^{\mu_n} = \inf_{i \in I} U^{\mu_i}$ quasi everywhere in Ω.

Remark. After this note was presented, G. Choquet has announced the same result as our Theorem 2 in his paper: Sur les fondements de la théorie fine du potentiel, C. R. Acad. Sci., Paris, **244**, 1606–1609 (1957).

References

[1] M. Brelot: Sur le potentiel et le suites de fonctions sousharmoniques, C. R. Acad. Sci., Paris, **207**, 836 (1938).
[2] M. Brelot: Nouvelle démonstration du théorème fondamental sur la convergence des potentiels, Ann. Inst. Fourier, **6**, 361–368 (1955–1956).
[3] M. Brelot and G. Choquet: Le théorème de convergence en théorie du potentiel (to appear).
[4] H. Cartan: Théorie du potential newtonien, énergie, capacité, suites de potentiels, Bull. Soc. Math., **73**, 74–106 (1945).
[5] G. Choquet: Theory of capacities, Ann. Inst. Fourier, **5**, 131–295 (1953–1954).
[6] G. Choquet: Les noyaux réguliers en théorie du potential, C. R. Acad. Sci., Paris, **243**, 635–638 (1956).
[7] M. Kishi: On a theorem of Ugaheri, Proc. Japan Acad., **32**, 314–319 (1956).
[8] N. Ninomiya: Sur le principe de continuité dans la théorie du potentiel, Jour. Inst. Polytech. Osaka City Univ., **8**, 51–56 (1957).
[9] M. Ohtsuka: Sur un espace complet de mesures positives dans la théorie du potentiel, Proc. Japan Acad., **32**, 311–313 (1956).
[10] M. Ohtsuka: Sur un théorème de M. Kishi, Proc. Japan Acad., **32**, 722–725 (1956).
[11] M. Ohtsuka: Les relations entre certains principes en théorie du potentiel, Proc. Japan Acad., **33**, 37–40 (1957).
[12] M. Ohtsuka: Selected Topics in the Theory of Functions, Tokyo (1957) (in Japanese).