77. Note on Orlicz-Birnbaum-Amemiya's Theorem

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W. Orlicz and Z. Birnbaum proved in [2] that an Orlicz space $L_\Phi(G)$ is finite if and only if the function Φ satisfies the following condition for some $\gamma>0$:

$$\Phi(2t)\leqq\gamma\Phi(t) \qquad \text{for every} \quad t\geqq t_0.$$

(In case of $mes(G)=+\infty$, $\Phi(2t)\leqq\gamma\Phi(t)$ for all $t\geqq 0$.)

This fact was generalized for arbitrary monotone complete modulars[1] on non-discrete spaces by I. Amemiya in [1] recently. In this note we shall show a new simple proof to this Amemiya's theorem.

As for an Orlicz-sequence space l_Φ, W. Orlicz and Z. Birnbaum also proved in [2] that l_Φ is finite if and only if the function Φ satisfies the following condition for some $\gamma>0$:

$$\Phi(2t)\leqq\gamma\Phi(t) \qquad \text{for every} \quad 0\leqq t\leqq t_0.$$

We shall generalize this fact on arbitrary modulars on discrete spaces.

§1. Let R be a universally continuous semi-ordered space and m be a modular on R. A modular is said to be "*finite*", if $m(x)<+\infty$ for every $x\in R$. And a modular on R is said to be "*semi-upper bounded*", if for every $\varepsilon>0$ there exists γ_ε ($\gamma_\varepsilon>0$) such that $m(x)\geqq\varepsilon$ implies $m(2x)\leqq\gamma_\varepsilon m(x)$. Now we shall prove

Theorem 1 (I. Amemiya). *Suppose that R has no atomic element, then every monotone complete finite modular on R is semi-upper bounded.*

Proof. We shall prove first that there exists γ_1 such that $m(x)\geqq 1$ implies $m(2x)\leqq\gamma_1 m(x)$. If such γ_1 can not be found, then we can find a sequence of elements $0\leqq x_\nu\in R$ ($\nu=1,2,\cdots$) such that

$$(1) \qquad m(2x_\nu)>\nu 2^{\nu+1}m(x_\nu),\ N_\nu\leqq m(x_\nu)\leqq N_\nu+1 \ (\nu=1,2,\cdots),$$

where N_ν ($\nu\geqq 1$) is a natural number.

(1) implies immediately

$$(2) \qquad m(2x_\nu)>\nu 2^\nu(N_\nu+1) \qquad (\nu=1,2,\cdots).$$

Since R has no atomic element, x_ν can be decomposed orthogonally as $x_\nu=\sum_{\mu=1}^{(N_\nu+1)2^\nu}x_{\nu,\mu}$, $m(x_{\nu,\mu})=m(x_{\nu,\rho})$ ($\mu,\rho=1,2,\cdots,(N_\nu+1)2^\nu$) for every $\nu\geqq 1$. As $m(x_\nu)<N_\nu+1$, we have $m(x_{\nu,\mu})\leqq\frac{1}{2^\nu}$ for every $1\leqq\mu\leqq 2^\nu(N_\nu+1)$.

1) For the definition of the modular see H. Nakano [3]. A modular m is said to be monotone complete, if $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}$, $\sup_{\lambda\in\Lambda}m(a_\lambda)<+\infty$ implies the existence of $\bigcup_{\lambda\in\Lambda}a_\lambda$.

If $m(2x_{\nu,\mu})\leqq\nu$ for each μ, we obtain

$$m(2x_\nu)=\sum_{\mu=1}^{(N_\nu+1)2^\nu} m(2x_{\nu,\mu})\leqq\nu 2^\nu(N_\nu+1),$$

which contradicts (2).

Therefore we can find a suffix μ_ν such that

$$(3)\qquad m(2x_{\nu,\mu_\nu})>\nu \qquad \text{for every} \quad \nu\geqq 1.$$

Putting $y_\rho=\bigcup_{\nu=1}^{\rho} x_{\nu,\mu_\nu}$, we obtain $0\leqq y_\rho \uparrow$ and $\sup_{\rho=1,2,\cdots} m(y_\rho)\leqq \sum_{\nu=1}^{\infty} m(x_{\nu,\mu_\nu})\leqq 1$. Since m is monotone complete by assumption, there exists $x_0\in R$ such that $x_0=\bigcup_{\rho=1}^{\infty} y_\rho$. For this x_0, however, we have by (3)

$$m(2x_0)\geqq m(2y_\nu)\geqq m(2x_{\nu,\mu_\nu})\geqq\nu \qquad \text{for every} \quad \nu\geqq 1.$$

This yields $m(2x_0)=\infty$ and contradicts that m is finite. Thus we showed that there exists γ_1 such that $m(x)\geqq 1$ implies $m(2x)\leqq\gamma_1 m(x)$.

For any ε $(1>\varepsilon>0)$, we set $\gamma_\varepsilon=\mathrm{Max}(\gamma_1, \frac{1}{\varepsilon}\sup_{\varepsilon\leqq m(x)\leqq 1} m(2x))$. Then it is easily seen that this γ_ε satisfies the condition of "*semi-upper bounded*". Thus the proof is completed.

§2. Here let R be a discrete semi-ordered linear space and e_λ $(\lambda\in\Lambda)$ a basis of R, i.e. $e_\lambda\frown e_\gamma=0$ for $\lambda\neq\gamma$ and for each positive element $x\in R$ we can find uniquely a system of real numbers $\xi_\lambda\geqq 0$ $(\lambda\in\Lambda)$ such that $x=\bigcup_{\lambda\in\Lambda}\xi_\lambda e_\lambda$.

Thus every $x\in R$ corresponds uniquely to a system of real numbers $(\xi_\lambda)_{\lambda\in\Lambda}$. We say an element $x=(\xi_\lambda)_{\lambda\in\Lambda}$ is finite dimensional, if $\xi_\lambda=0$ except for finite numbers of $\lambda\in\Lambda$.

Let m be a modular on R. Putting $\varphi_\lambda(\xi)=m(\xi e_\lambda)$, we obtain a modular $\varphi_\lambda(\xi)$ $(\lambda\in\Lambda)$ on the space of real numbers, that is, i) $\varphi_\lambda(0)=0$; ii) $\lim_{\xi\to\eta-0}\varphi_\lambda(\xi)=\varphi_\lambda(\eta)$; iii) $\lim_{\xi\to+\infty}\varphi_\lambda(\xi)=+\infty$; iv) there exists a real number η (depending on each φ_λ) such that $\varphi_\lambda(\eta)<+\infty$ for every $\lambda\in\Lambda$.

Conversely if $\varphi_\lambda(\xi)$ satisfies the above conditions for every $\lambda\in\Lambda$, then the set of such systems of real numbers $(\xi_\lambda)_{\lambda\in\Lambda}$ that

$$\sum_{\lambda\in\Lambda}\varphi_\lambda(\alpha\xi_\lambda)<+\infty \qquad \text{for some } \alpha>0$$

becomes a discrete modulared space, putting its modular as

$$m(x)=\sum_{\lambda\in\Lambda}\varphi_\lambda(\xi_\lambda) \qquad \text{for} \quad x=(\xi_\lambda)_{\lambda\in\Lambda}.$$

And we denote this discrete modulared space by $l(\varphi_\lambda)_{\lambda\in\Lambda}$. This modulared space is always monotone complete.

A modular m is said to be simple if $m(x)=0$ implies $x=0$. From the above, we can see that a modular m on discrete space R is simple if and only if $\varphi_\lambda(\xi)>0$ for every $\xi>0$ and $\lambda\in\Lambda$.

If $\varphi_\lambda(\xi)$ is equal to a single function $\varphi_0(\xi)$ for every $\lambda\in\Lambda$, then this modular is said to be *constant* (cf. [3, §55]), and it is nothing but a generalized Orlicz discrete space. A constant modular m on R is finite if and only if $\varphi_0(2\xi)\leqq\gamma\varphi_0(\xi)$ $(0\leqq\xi\leqq\xi_0)$ for some $\gamma>0$ and $\xi_0>0$ that is, $m(2x)\leqq\varepsilon$ implies $m(2x)\leqq\gamma m(x)$ (cf. [3]).

This fact, however, is not valid for arbitrary modular on an infinite dimensional discrete space, even if it is simple. The example will be showed in the following.

Theorem 2. *Let m be a monotone complete modular on a discrete space R. In order that m is finite it is necessary and sufficient that it satisfies the following conditions:*

i) $\varphi_\lambda(\xi)<+\infty$ *for every* $\xi\geqq 0$ *and* $\lambda\in\Lambda$;

ii) *there exist positive numbers* ε *and* ε' $(0<\varepsilon<\varepsilon')$ *such that* $\varepsilon\leqq m(x)\leqq\varepsilon'$ *implies* $m(2x)\leqq\gamma m(x)$ *for some* $\gamma>0$.

Proof. *Necessity.* Let m be finite. Then i) is obvious because of $\varphi_\lambda(\xi)=m(\xi e_\lambda)$ $(\lambda\in\Lambda)$. In order to prove ii), we suppose that ii) fails to be true. Then we construct consecutively an orthogonal sequence of elements $0\leqq x_\nu\in R$ $(\nu=1,2,\cdots)$ such that $\frac{1}{2^{\nu+1}}\leqq m(x_\nu)\leqq\frac{1}{2^\nu}$, $m(2x_\nu)\geqq 2^\nu m(x_\nu)$ and x_ν is finite dimensional for every $\nu\geqq 1$. Suppose that $x_1,x_2,\cdots,x_n$ had been taken already. Since $[x_1,\cdots,x_n]R$[2] is finite dimensional, we can find a positive number γ' such that $\frac{1}{2^{\nu+2}}\leqq m(x)\leqq\frac{1}{2^{\nu+1}}$, $x\in[x_1,\cdots,x_n]R$ implies $m(2x)\leqq\gamma' m(x)$ by virtue of i). If there exists positive number γ'' such that $\frac{1}{2^{\nu+2}}<m(x)\leqq\frac{1}{2^{\nu+1}}$, $x\in(1-[x_1,\cdots,x_n])R$ implies $m(2x)\leqq\gamma'' m(x)$, then ii) holds true for $\varepsilon=\frac{1}{2^{\nu+2}}$, $\varepsilon'=\frac{1}{2^{\nu+1}}$ and $\gamma=\gamma'+\gamma''$.

Therefore we can find $0\leqq x\in(1-[x_1,\cdots,x_n])R$ which satisfies

$$\frac{1}{2^{\nu+2}}<m(x)\leqq\frac{1}{2^{\nu+1}},\qquad m(2x)>2^{\nu+1}m(x).$$

Since m is semi-continuous, there exists $y\in R$ $(0\leqq y\leqq x)$ such that $\frac{1}{2^{\nu+2}}\leqq m(y)\leqq\frac{1}{2^{\nu+1}}$, $m(2y)>2^{\nu+1}m(y)$ and y is finite dimensional. Here we obtain $x_{\nu+1}$, putting $x_{\nu+1}=y$. For such a sequence $\{x_\nu\}$ we have $m(\bigcup_{\nu=1}^{n}x_\nu)\leqq 1$. Thus there exists $\bigcup_{\nu=1}^{\infty}x_\nu=x_0$. However, we have for this x_0

$$m(2x_0)\geqq\sum_{\nu=1}^{\infty}2^\nu\frac{1}{2^{\nu+1}}=+\infty,$$

which contradicts that m is finite on R. Thus ii) is proved.

Sufficiency. Let x be an element of R such that $m(x)<\infty$. Then we can decompose x as $x=y+z$, such that $m(y)\leqq\varepsilon'$ and z is finite dimensional. i) implies $m(2z)<+\infty$ and existence of y_0 such that $y_0\geqq|y|$ and $\varepsilon\leqq m(y_0)\leqq\varepsilon'$. Then we have by ii) $m(2y_0)\leqq\gamma m(y_0)$, and $m(2x)$

2) For a set $A\subset R$, $[A]R$ means the least normal manifold containing A.

$\leqq m(2z)+m(2y_0)<+\infty$. Thus we obtain that m is finite.

Remark 1. In the above theorem ε (which appears in the condition ii)) can be taken arbitrary small by varying γ, but ε' can not be.

Remark 2. In the above theorem, the assumption: "$\varepsilon\leqq m(x)\leqq\varepsilon'$" can not be replaced by "$0<m(x)\leqq\varepsilon'$" even if m is simple.

For example, set

$$\varphi_\nu(\xi)=\begin{cases}\dfrac{1}{2^\nu}\xi & \text{for } 0\leqq\xi\leqq\dfrac{1}{2^\nu},\\ \left(\dfrac{2^{\nu+1}-1}{2\nu}\right)\left(\xi-\dfrac{1}{2^\nu}\right)+\dfrac{1}{2^{2\nu}} & \text{for } \dfrac{1}{2^\nu}\leqq\xi\leqq\dfrac{1}{2^{\nu-1}},\\ 2\xi-\dfrac{1}{2^{\nu-1}} & \text{for } \dfrac{1}{2^{\nu-1}}\leqq\xi,\end{cases}$$

and consider modulared sequence space $l(\varphi_1, \varphi_2,\cdots)$. Then $l(\varphi_1, \varphi_2,\cdots)$ is finite and simple as easily seen. On the other hand, putting $c_n=\frac{1}{2^n}e_n$, where e_n are the natural bases on sequence spaces, we have

$$(2c_n)\geqq 2^n m(c_n) \quad\text{and}\quad m(c_n)\leqq\frac{1}{2^{2n}}.$$

Thus the example is established.

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References

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