142. On Yamamuro's Theorem Concerned with Linear Modulars

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Some years ago S. Yamamuro [1] proved the following theorem which is the first success in the special case of the so-called norm problem in the theory of modulared semi-ordered linear spaces inaugurated by H. Nakano [2].

Theorem. Let R be a semi-regular modulared semi-ordered linear space with a modular m. In order that the first norm $\|a\|$ and the second norm $|||a|||$ defined by m coincide for every $a\in R$, it is necessary and sufficient that the modular m is either linear or singular.

Since then, I. Amemiya obtained the following convenient formula for the first norm:

$$(1) \qquad \|a\|=\inf_{\xi>0}\frac{1+m(\xi a)}{\xi},$$

and, by systematic uses of this formula, he established a characterization of the modulars of L_p type ($p>1$) in [3] which, together with above Yamamuro's theorem, constitutes a sufficiently general solution of the norm problem.

The purpose of this short note is to show that Amemiya's method is also applicable for the above theorem and we can obtain its extremely simple proof without use of conjugate modulars.

Since the sufficiency is well known (cf. [2, Theorems 41.3 and 41.4]), we shall prove the necessity dividing in three steps in follows. In the sequel, we suppose that R is a semi-regular modulared semi-ordered linear space in which the two norms defined by its modular always coincide.

1. *For any $a\in R$ such that $0<m(a)<+\infty$ there exists a real number $0<\alpha<+\infty$ for which we have $m(\alpha a)=1$.*

Since the above statement is evidently true by modular conditions if there exists a real number $0<\beta<+\infty$ such that $1\leqq m(\beta a)<+\infty$, we suppose that

$$m(\xi a)<1 \text{ for } 0\leqq\xi\leqq\gamma, \text{ and } m(\xi a)=+\infty \text{ for } \xi>\gamma$$

for some real number $0<\gamma<+\infty$. Then, by the formula (1), there exists a real number $0<\xi_0<+\infty$ for which we have

$$0<\xi_0\leqq\gamma, \quad \|a\|=\frac{1+m(\xi_0 a)}{\xi_0}.$$

On the other hand we have

$$|||a||| = \inf_{\xi>0,\, m(\xi a)\leqq 1} \frac{1}{\xi} = \frac{1}{\gamma},$$

and hence, by the coincidence of two norms,

$$0<\xi_0 \leqq \gamma, \quad \frac{1+m(\xi_0 a)}{\xi_0} = \frac{1}{\gamma}.$$

This implies that $m(\xi_0 a)=0$ and $\xi_0=\gamma$, and therefore we have $m(a)=0$ because $\gamma \geqq 1$ from $m(a)<+\infty$, which contradicts with $m(a)>0$.

2. *Every element $a \in R$ such that $m(a)=1$ is a linear element, namely*

$$m(\xi a) = \xi m(a)$$

for any real number $0 \leqq \xi < +\infty$.

From $m(a)=1$ we see easily $|||a|||=1$, and hence, by the coincidence of two norms,

$$\inf_{\xi>0} \frac{1+m(\xi a)}{\xi} = ||a|| = 1.$$

On the other hand, also from $m(a)=1$, we have

$$\frac{1+m(\xi a)}{\xi} > 1 \quad \text{for any } 0<\xi<+\infty,$$

and hence

$$(2) \qquad \lim_{\xi\to+\infty} \frac{1+m(\xi a)}{\xi} = \lim_{\xi\to+\infty} \pi(\xi \,|\, a) = 1,$$

where $\pi(\xi \,|\, a)$ denotes the right hand derivative of $m(\xi a)$ as a function of $0 \leqq \xi < +\infty$ (for this elementary fact (2), cf. [4]). Since $\pi(\xi \,|\, a)$ is a non-decreasing function of $0 \leqq \xi < +\infty$, this implies that

$$\pi(\xi \,|\, a) \leqq 1 \quad \text{for any } 0 \leqq \xi < +\infty,$$

and hence we have

$$m(\xi a) = \int_0^{\xi} \pi(\xi \,|\, a)\, d\xi \leqq \xi \quad \text{for any } 0 \leqq \xi < +\infty.$$

Now, by the convex character of the modular function $m(\xi a)$, it is clear that

$$m(\xi a) \geqq \xi m(a) = \xi \quad \text{for any } 1 \leqq \xi < +\infty,$$

and hence we have $m(\xi a)=\xi$ for any $1 \leqq \xi < +\infty$. Furthermore, by the convexity of $m(\xi a)$ again, we can conclude that $m(\xi a)=\xi$ holds also for $0 \leqq \xi < 1$.

3. *If there exists at least one element $a \in R$ such that $0<m(a)<+\infty$, then the modular m is linear.* (*If otherwise m is evidently singular.*)

Since an element $a \in R$ such that $0<m(a)<+\infty$ is a non-zero linear element (by the above two steps), denoting by N the totality of all linear elements in R, we obtain a normal manifold N of R different from $\{0\}$. Then the modular m is linear in N, its orthogonal complement $N^{\perp}$ has no linear element except 0 and hence m is singular in $N^{\perp}$ (by the above two steps). Now, if we have $N^{\perp} \neq \{0\}$, there exist at least one pair a, b such that $N \ni a > 0$ and $N^{\perp} \ni b > 0$, and by easy calculations we have

$$\| a+b \| = \| a \| + \| b \|, \quad ||| a+b ||| = \mathrm{Max}\{||| a |||, ||| b |||\}.$$

This contradicts the coincidence of two norms, and hence we have $N^{\perp} = \{0\}$, namely the modular m is linear in R.

References

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[4] N. Bourbaki: Fonctions d'une Variable Réelle (Théorie Élémentaire), Chap. 1, Exercices.