44. Decomposition-equivalence and the Existence of Non-measurable Sets in a Locally Compact Group

By Yoshimichi MIBU

(Comm. by Z. SUETUNA, M.J.A., April 12, 1958)

Let G be a locally compact and σ-compact group and m^* a left invariant outer measure in G. In the theory of Haar's measure, it is well known that any two measurable sets $A\subseteq G$ and $B\subseteq G$ of the same measure are decomposition-equivalent to each other, that is, there exist direct decompositions

$$(1)\qquad \begin{aligned} A&=M+A_1+A_2+\cdots+A_n+\cdots,\\ B&=N+B_1+B_2+\cdots+B_n+\cdots \end{aligned}$$

of A and B, with relations

$$(2)\qquad m^*(M)=m^*(N)=0,\ \ g_iA_i=B_i,\ \ g_i\in G,\ \ i=1,2,\cdots,$$

and

(3) every A_i is m^*-measurable.

Conversely, any two measurable sets which are decomposition-equivalent to each other have clearly the same measure. Hence if $m(A)\neq m(B)$ (for measurable set A we write $m(A)$ instead of $m^*(A)$), then the set A is not decomposition-equivalent to B. But if we admit, in the expression (1), non-measurable sets A_i's, then it is proved that for any two measurable sets A and B of positive measures, even though $m(A)$ is not equal to $m(B)$, there exist direct decompositions (1) satisfying the condition (2). This is included in the Corollary of Theorem 1 as a special case.

Definition. Let A and B be two subsets of G. If there exist direct decompositions (1) satisfying the condition (2), then A is called to be almost decomposition-equivalent to B. And if further in the expression (1) both M and N can be taken to be empty, A is called to be completely decomposition-equivalent to B.

Remark 1. In the above definition it is not assumed that each A_i is measurable. Our definition of decomposition-equivalence is different from the usual one.

Notation. In the following we denote by $A\sim B$ and $A\approx B$ the almost and completely decomposition-equivalence of A to B respectively.

Remark 2. Suppose that $A\approx B$. Then $m^*(A)=0$ implies $m^*(B)=0$. And if A is of the first category, then B is also of the same category.

The following lemma is easily proved.

Lemma 1. 1) $A\sim A\,(A\approx A)$, 2) $A\sim B\,(A\approx B)$ implies $B\sim A\,(B\approx A)$, 3) $A\sim B$ $(A\approx B)$ and $B\sim D$ $(B\approx D)$ imply $A\sim D$ $(A\approx D)$.

Lemma 2. Suppose that $A\approx B$ $(A\sim B)$ and $B\subseteq A$. If $B\subseteq D\subseteq A$, then $A\approx D$ $(A\sim D)$.

Proof. By the assumption of our theorem, there exist direct decompositions $A=A_1+A_2+\cdots+A_n+\cdots$ and $B=B_1+B_2+\cdots+B_n+\cdots$ such that $g_iA_i=B_i$, $i=1,2,\cdots$. For any $x\in A$, there exists an A_i such that $x\in A_i$. We define $f(x)=g_ix$ if $x\in A_i$. Under such definition of $f(x)$, it becomes a one-to-one mapping of A onto B and $f(A_i)=B_i$, $i=1,2,\cdots$ hold. Moreover it is easily seen that

(4) $E\approx f(E)$ for any subset E of A.

We set $A-D=K$ and $D-B=L$. It clearly holds that $A=B+K+L$. In the rest of the present proof, we shall write $E_1=f(E)$, $E_2=f(E_1)$, $\cdots E_{i+1}=f(E_i)\cdots$ for any subset $E\subseteq A$. In such notation we have

$$B=f(A)=f(B)+f(K)+f(L)=B_1+K_1+L_1$$
$$B_1=f(B)=f(B_1)+f(K_1)+f(L_1)=B_2+K_2+L_2$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$B_i=f(B_{i-1})=f(B_i)+f(K_i)+f(L_i)=B_{i+1}+K_{i+1}+L_{i+1}$$
$$\cdots\cdots$$
$$\cdots\cdots\ .$$

Let $B^*=\bigcap_{i=1}^{\infty}B_i$. We have $A=B^*+K+L+K_1+L_1+\cdots$ and $D=B^*+L+K_1+L_1+\cdots$. On the other hand from the above relation (4) we have $K\approx K_1\approx K_2\approx\cdots$, $L\approx L_1\approx L_2\approx\cdots$. Then it is easily seen that $A\approx D$.

When $A\sim B$ and $B\subseteq D\subseteq A$ we can easily prove that $A\sim D$ by a slight modification of the above proof.

Lemma 3. Let H be an abstract subgroup of G such that $\overline{\overline{H}}\leq\aleph_0$.

If $G=H\alpha+H\beta+H\gamma+\cdots$ and $G=H\alpha'+H\beta'+H\gamma'+\cdots$ are two expressions of the decomposition of G into the right cosets of H, then the set $M=\{\alpha,\beta,\gamma,\cdots\}$ is completely decomposition-equivalent to the set $M'=\{\alpha',\beta',\gamma',\cdots\}$, that is, $M\approx M'$.

Proof. Without losing the generality we may assume that $H\alpha=H\alpha'$, $H\beta=H\beta'$, $H\gamma=H\gamma',\cdots$. This assumption means that there exist elements $\xi,\eta,\zeta,\cdots$ of H such that $\alpha'=\xi\alpha$, $\beta'=\eta\beta$, $\gamma'=\zeta\gamma,\cdots$. By putting for each element $h\in H$

$$M'_h=\{\alpha';\alpha'=h\alpha,\alpha\in M\}\frown M',\qquad M_h=\{\alpha;\alpha'=h\alpha,\alpha'\in M'\}\frown M,$$

we have clearly $M'_h=hM_h$. On the other hand $M=\sum_{h\in H}M_h$ and $M'=\sum_{h\in H}M'_h$ hold. Hence we have clearly $M\approx M'$.

Theorem 1. Let G be a locally compact and σ-compact group and A a subset of G. If G is not discrete, then it follows that

1) if A contains a measurable subset E of positive measure, then $A\sim G$,

2) if A^i (the set of inner points of A) is not empty, then $A\approx G$.

Proof. In the first place we shall prove 1). Since $m(E)>0$, there exists a sequence $a_1, a_2, \cdots, a_n, \cdots$ of elements of G such that $m^*(G - \bigcup_{i=1}^{\infty} a_i E) = 0$.

This is the well-known fact in the theory of Haar's measure. By putting $B = G - \bigcup_{i=1}^{\infty} a_i E$ and $D = E \smile B$, we see at once

$$G = \bigcup_{i=0}^{\infty} a_i D \quad \text{where } a_0 = e. \tag{5}$$

Let H be the abstract subgroup of G which is generated by $\{a_0, a_1, \cdots, a_n, \cdots\}$. Then we have clearly

$$\overline{\overline{H}} \leq \aleph_0 \quad \text{and} \quad HD = G. \tag{6}$$

G is partitioned into the right cosets of H such that

$$G = H\alpha + H\beta + H\gamma + \cdots \ . \tag{7}$$

From (6) we can assume that $\alpha, \beta, \gamma, \cdots$ are all contained in D. We set $L = \{\alpha, \beta, \gamma, \cdots\} \subseteq D$. Since G is locally compact but not discrete, we have easily $\overline{\overline{G}} > \aleph_0$. Hence there exists an element $g_1 \in G$ such that $g_1 \bar{\in} H$. Let H_1 be the abstract subgroup of G which is generated by H and g_1. Then clearly $\overline{\overline{H}}_1 \leq \aleph_0$, and hence there exists an element g_2 such that $g_2 \bar{\in} H_1$. Let H_2 be the abstract group generated by H_1 and g_2. Then we have also $\overline{\overline{H}}_2 \leq \aleph_0$. Continuing this process we have finally

$$H \subset H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots \ . \tag{8}$$

Let H^* be the abstract subgroup of G which is generated by $H, H_1, H_2, \cdots$. Then it is easily seen that $\overline{\overline{H}}^* = \aleph_0$. H^* is decomposed into the right cosets of H such that

$$H^* = H\lambda + H\mu + H\nu + \cdots \ . \tag{9}$$

Denoting $\{\lambda, \mu, \nu, \cdots\}$ by M, we have clearly $\overline{\overline{M}} = \aleph_0$. On the other hand G is decomposed into the right cosets of H^* such that

$$G = H^*\xi + H^*\eta + H^*\zeta + \cdots \ . \tag{10}$$

We set $N = \{\xi, \eta, \zeta, \cdots\}$. From (9) and (10) we have

$$\begin{array}{rl} G = & H\lambda\xi + H\mu\xi + H\nu\xi + \cdots \\ & + H\lambda\eta + H\mu\eta + H\nu\eta + \cdots \\ & + H\lambda\zeta + H\mu\zeta + H\nu\zeta + \cdots \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad . \end{array} \tag{11}$$

Setting

$$K = \left\{ \begin{array}{llll} \lambda\xi, & \mu\xi, & \nu\xi, \cdots \\ \lambda\eta, & \mu\eta, & \nu\eta, \cdots \\ \lambda\zeta, & \mu\zeta, & \nu\zeta, \cdots \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \end{array} \right\},$$

we have $K = \lambda\{\xi, \eta, \zeta, \cdots\} + \mu\{\xi, \eta, \zeta, \cdots\} + \nu\{\xi, \eta, \zeta, \cdots\} + \cdots = \lambda N + \mu N$

$+\nu N+\cdots=\sum_{\sigma\in M}\sigma N$, that is,

$$K=\sum_{\sigma\in M}\sigma N. \tag{12}$$

Using Lemma 3 we have $L\approx K$. We shall show that $K\approx G$. From (11) we have $G=\sum_{h\in H}hK=\sum_{h\in H}h(\sum_{\sigma\in M}\sigma N)=\sum_{h\in H,\sigma\in M}h\sigma N$, i.e.,

$$G=\sum_{h\in H,\sigma\in M}h\sigma N. \tag{13}$$

Since $\overline{\overline{M}}=\aleph_0=\overline{\overline{H\cdot M}}$, we see at once $K\approx G$. Hence by Lemma 1 we have $L\approx G$. From Lemma 2 we have $D\approx G$. Since D is the sum of the set E and the set B of measure 0, it is easily seen that $E\sim G$ (see Remark 2). Using again Lemma 2 we have $A\sim G$. This completes the proof of 1).

For the proof of 2) we select a measurable open set E such that $E\subseteq A^i$. Then in the proof of assertion 1) the set B can be taken to be empty, and hence $D=E$. Consequently we have $E\approx G$. From Lemma 2 we have $A\approx G$.

Corollary. Let A and B be two subsets of G.

1) If each of A and B contains its measurable subset of positive measure, then $A\sim B$ and

2) if $A^i\neq 0$, $B^i\neq 0$, then $A\approx B$.

Theorem 2. Let G be a locally compact and σ-compact group. If G is not discrete, then there exists a non-measurable set in G. More generally, any measurable set of positive measure contains a non-measurable set.

Proof. Let A be a measurable set of positive measure. It is easily seen that there exists a measurable subset $B\subseteq A$ such that $0<m(B)\neq m(G)$. Then from the above theorem we have $B\sim G$. Consequently $m(B)=m(G)$ if every subset of B is measurable. So we have arrived at a contradiction.

Remark 3. The first half of the above theorem holds for any non-σ-compact group G. This is evident from the fact that G contains an open and σ-compact subgroup.

Remark 4. Let G be a separable and locally compact group. Then there exists a set which is almost decomposition-equivalent to G but not completely decomposition-equivalent to G.

Proof. It is easily seen that G contains a measurable non-dense set A of positive measure. From Theorem 1 we have $A\sim G$. Since G is of the second category, A is not completely decomposition-equivalent to G (see Remark 2).