79. On the Relations "Semi-between" and "Parallel" in Lattices

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In a recent paper [2], we have studied the concept of B-covers in lattices as a generalization of the metric-betweeness in a normed lattice which is investigated by L. M. Kelley [1], and discussed some geometrical properties of lattices by means of B-covers and B^*-covers in [3, 4].

At first we shall introduce the concept of J-cover and CJ-cover which will be considered as semi-between sets in lattices since the B-covers are treated as between sets in lattices. For any two elements a and b of a lattice L, we shall define as follows:

$J(a,b)=\{x\mid(a\frown x)\smile(b\frown x)=x\}$, $CJ(a,b)=\{x\mid(a\smile x)\frown(b\smile x)=x\}$.

$J(a,b)$ is called the J-cover of a and b, and if $x\in J(a,b)$, then we shall write $J(axb)$. Similarly we shall define CJ-cover and $CJ(axb)$. Further, we define $J^*(a,b)=\{x\mid J(abx)\}$, $CJ^*(a,b)=\{x\mid CJ(abx)\}$, $J(a,J^*(a,b))=\{y\mid J(ayx)$ for all $x\in J^*(a,b)\}$, etc.

$B(a,b)=J(a,b)\frown CJ(a,b)$ is called the B-cover of a and b; and we write axb when $x\in B(a,b)$ (cf. [2–4]).

Next we shall define the notion of "parallel" as follows: $ab//cd$ means that $B^*(a,b)\frown B^*(c,d)=0$, where $B^*(a,b)=\{x\mid abx\}$.

In §1 we shall give characterizations of modular or distributive lattices by means of "semi-between", and in §2 we shall consider the geometrical properties of lattice polygons by the notion of "*parallel*".

§1. "Semi-between". Lemma 1. *$(a]\smile(b]=J(a,b)\subset(a\smile b]$, $[a)\frown[b)=CJ(a,b)\subset[a\frown b)$, where $(x]=\{z\mid z\leqq x\}$, $[x)=\{z\mid z\geqq x\}$, $A\smile B=\{x\smile y\mid x\in A,\ y\in B\}$ if $A,B\subset L$.*

The proof is found in [2].

Lemma 2. *$J(axb)$ implies $x\frown(a\smile b)=(a\frown x)\smile(b\frown x)=x$; $CJ(axb)$ implies $x\smile(a\frown b)=(a\smile x)\frown(b\smile x)=x$.*

Lemma 3. *$J(abc)$, $CJ(axb)$ and $CJ(byc)$ imply $J(xby)$. $CJ(abc)$, $J(axb)$ and $J(byc)$ imply $CJ(xby)$.*

Proof. We have $b\geqq b\frown x\geqq a\frown b$, $b\geqq b\frown y\geqq b\frown c$ by $CJ(axb)$, $CJ(byc)$, and hence $b\geqq(b\frown x)\smile(b\frown y)\geqq(a\frown b)\smile(b\frown c)=b$ by $J(abc)$; thus we have $(b\frown x)\smile(b\frown y)=b$, that is, $J(xby)$.

Lemma 4. *$J(axb)$ and $J(ayb)$ imply $J(a(x\smile y)b)$. $CJ(axb)$ and $CJ(ayb)$ imply $CJ(a(x\frown y)b)$. $J(axb)$ and $J(ayb)$ do not necessarily imply $J(a(x\frown y)b)$.*

Proof. Since $x\leqq a\smile b$, $y\leqq a\smile b$ by $J(axb)$, $J(ayb)$, we have $x\smile y = (a\smile b)\frown(x\smile y)\geqq(a\frown(x\smile y))\smile(b\frown(x\smile y))\geqq(a\frown x)\smile(a\frown y)\smile(b\frown x)\smile(b\frown y)=x\smile y$, and hence we have $(a\frown(x\smile y))\smile(b\frown(x\smile y))=x\smile y$, that is, $J(a(x\smile y)b)$. If L contains elements $a,b,x,y,z,\ z_1,x_1,y_1$, such that $a\smile b>x_1>a$, $a\smile b>y_1>b$, $a>x>a\frown b$, $b>y>a\frown b$, $x_1\smile y_1=a\smile b$, $x_1\frown y_1=z_1>x\smile y=z$, $x\frown y=a\frown b$, $a\smile z=a\smile z_1=x_1$, $b\smile z=b\smile z_1=y_1$, $a\frown z=a\frown z_1=x$, $b\frown z=b\frown z_1=y$, then we have $J(ax_1b)$, $J(ay_1b)$ but not $J(a(x_1\frown y_1)b)$.

Lemma 5. *In case L is modular, $J(axb)$ implies $(a\smile x)\frown(b\smile x)=x\smile(a\frown b)$.*

Proof. If $J(axb)$, then we have by modularity $(a\smile x)\frown(b\smile x)=x\smile(a\frown(b\smile x))=x\smile(a\frown((a\frown x)\smile(b\frown x)\smile b))=x\smile(a\frown((a\frown x)\smile b))=x\smile(a\frown x)\smile(a\frown b)=x\smile(a\frown b)$.

Lemma 6. *In case L is modular, if $x\in B(a\smile b,\ a\frown b)$, then $J(axb)$ implies $CJ(axb)$ and $CJ(axb)$ implies $J(axb)$.*

Theorem 1.1. *In order that L is a modular lattice it is necessary and sufficient that $J(axb)$ implies $(a\smile x)\frown(b\smile x)=x\smile(a\frown b)$.*

Proof. If L is not modular, then there exist five elements a,b,c,d,x such that $c=a\smile b=x\smile b$, $d=a\frown b=x\frown b$, $d<x<a<c$. In this case we have $J(axb)$ but $(a\smile x)\frown(b\smile x)=a>x=x\smile(a\frown b)$. By Lemma 5 this completes the proof.

Theorem 1.2. *In case L is modular, if $x,y\in B(a\smile b,\ a\frown b)$, then $J(axb)$ and $J(ayb)$ imply $a(x\smile y)b$, $a(x\frown y)b$.*

Proof. It is obvious from Lemmas 4 and 6.

Lemma 7. *$J(abc)$ implies $a(a\smile b)c$. $CJ(abc)$ implies $a(a\frown b)c$.*

Proof. We have $b=(a\frown b)\smile(b\frown c)\leqq a\smile(b\frown c)$, $a\leqq a\smile(b\frown c)$, and hence $a\smile b\leqq a\smile(b\frown c)\leqq a\smile b$, thus we have $a\smile(b\frown c)=a\smile b$.

Accordingly we have $a\smile b=a\smile(b\frown c)=a\smile(a\frown c)\smile(b\frown c)\leqq a\smile((a\smile b)\frown c)\leqq a\smile b\smile((a\smile b)\frown c)=a\smile b$, hence $a\smile((a\smile b)\frown c)=a\smile b$, that is, $J(a(a\smile b)c)$. Since $CJ(a(a\smile b)c)$ is trivial we have $a(a\smile b)c$.

Lemma 8. *$J(abc)$ implies $J((a\frown b)bc)$ and vice versa.*

Lemma 9. *In order that L is a distributive lattice it is necessary and sufficient that the condition* (D) *below holds for any elements a,b of L.*

(D) *$x\in J(a,b)$ if and only if $x\leqq a\smile b$.*

Proof. It is proved in the same way as Theorem 3 [2].

Theorem 1.3. *In any lattice we have the following inequalities:*

① $J^*(a,b)\subset J^*(a,a\smile b)$, $CJ^*(a,b)\subset CJ^*(a,a\frown b)$,

② $J^*(a\frown b,a\smile b)\subset J^*(a,a\smile b)\frown J^*(b,a\smile b)$, $CJ^*(a\smile b,a\frown b)\subset CJ^*(a,a\frown b)\frown CJ^*(b,a\frown b)$.

We have the equalities in ①, ② for a distributive lattice.

Proof. The proof of ① is obtained by Lemma 7.

② Since $CJ((a\frown b)\,a\,(a\smile b))$ and $CJ((a\frown b)\,b\,(a\smile b))$ are trivial, suppose that $J((a\frown b)(a\smile b)x)$; then we have $J(a(a\smile b)x)$, $J(b(a\smile b)x)$ by Lemma 3. In case L is distributive, for ② if we take $x\in J^*(a,a\smile b)\frown J^*(b,a\smile b)$, then we have $a\smile b\leqq a\smile x$, $a\smile b\leqq b\smile x$ by Lemma 2, and hence $a\smile b\leqq(a\smile x)\frown(b\smile x)=x\smile(a\frown b)$ by distributive law, thus we have $(a\frown b)(a\smile b)x$ by Lemma 9.

Similarly we may prove the remaining parts.

Theorem 1.4. *In any lattice we have the following equalities:*

① $J^*(a,b)=J^*(a\frown b,b)$, $CJ^*(a,b)=CJ^*(a\smile b,b)$,

② $J^*(CJ(a,b),b)=J^*(a,b)$, $CJ^*(J(a,b),b)=CJ^*(a,b)$,

③ $J^*(CJ(a,J^*(a,b)),a)=J^*(b,a)$.

Proof. ① It is obvious by Lemma 8.

② Since it is trivial that $J^*(CJ(a,b),b)\subset J^*(a,b)$, we shall prove the inverse relation. If we take x from $J^*(a,b)$, then for any $y\in CJ(a,b)$ we have $J(ybx)$ by Lemma 3, and hence x belongs to $J^*(CJ(a,b),b)$.

③ Since $CJ(a,b)\supset CJ(a,J^*(a,b))\ni b$, and $J^*(b,a)=J^*(CJ(a,b),a)$ by ②, we have $J^*(b,a)=J^*(CJ(a,b),a)\subset J^*(CJ(a,J^*(a,b)),a)\subset J^*(b,a)$, and hence we have the equality of ③.

Lemma 10. *In any lattice $J(a,b)=J(c,d)$ implies $a\smile b=c\smile d$, and $CJ(a,b)=CJ(c,d)$ implies $a\frown b=c\frown d$.*

We have the converse of Lemma 10 in a distributive lattice.

Theorem 1.5. *In a lattice L suppose that $J^*(a,B(a,a\smile b))=a\smile b$, then ① $a\smile b$ is a maximal element,*

② if a and b are non-comparable, then there exists at least one element x in L such that $a<x<a\smile b$ and x does not belong to $J(a,b)$.

It is proved in the same way as (1), §4 [5].

Lemma 11. *$CJ(axb)$ for $x\in B(b,a\frown b)$ implies $a\smile x\in J(a,b)$; $J(ayb)$ for $y\in B(a,a\smile b)$ implies $b\frown y\in CJ(a,b)$.*

Proof. Suppose that $CJ(axb)$ for $x\in B(b,a\frown b)$; then we have $x=(a\smile x)\frown(b\smile x)=(a\smile x)\frown b$, and hence we have $(a\frown(a\smile x))\smile((a\smile x)\frown b)=a\smile x$, that is, $J(a(a\smile x)b)$. Similarly we have the other part.

Theorem 1.6. *Suppose that $CJ(a,b)\supset B(b,a\frown b)$ and $J(a,b)\supset B(a,a\smile b)$ in a lattice L; then $B(a,a\smile b)$ is isomorphic to $B(b,a\frown b)$.*

Proof. For $x_i\in CJ(a,b)$ where $a\frown b\leqq x_i\leqq b$, $i=1,2$, we have $b\frown(a\smile x_i)=(b\smile x_i)\frown(a\smile x_i)=x_i$. Similarly we have $(b\frown y_j)\smile a=(b\frown y_j)\smile(a\frown y_j)=y_j$ for $a\leqq y_j\leqq a\smile b$, $y_j\in J(a,b)$, $j=1,2$. Hence if $a\smile x_1=a\smile x_2$, then we have $x_1=x_2$.

These mappings $x_i\to a\smile x_i$, $y_j\to b\frown y_j$ preserve order and are inverses of each other.

Lemma 12. $J(aIb)$ implies $a\smile b=I$, $CJ(aOb)$ implies $a\frown b=0$.

Theorem 1.7. *Suppose that $b\in J^*(a,I)\frown CJ^*(a,O)$; then we have $J^*(a,b)=[b)$, $CJ^*(a,b)=(b]$.*

Proof. If $b\in J^*(a,I)\frown CJ^*(a,O)$, then we have $a\smile b=I$, $a\frown b=O$

by Lemma 12, and hence by Theorem 1.4 we have $J^*(a,b)=J^*(a\frown b,b)$ $=J^*(O,b)=[b)$ since $b\frown x=b$ from $(O\frown b)\smile(b\frown x)=b$. Similarly we have $CJ^*(a,b)=CJ^*(a\smile b,b)=CJ^*(I,b)=(b]$.

§2. "Parallel". Henceforth we shall consider a lattice polygon as a sublattice of L, accordingly the notion of parallel is considered for all elements in L. For elements a,b,c,d of L, by "*a quad-rangle abcd*" we shall mean that a,b are non-comparable and $a\smile b=c$, $a\frown b=d$.

Theorem 2.1. *In a lattice quad-rangle abcd we have*

① $$cb//da,\ ca//db,$$

② $$cd//db.$$

Proof. ① Suppose that cbx, dax; then $a=(d\frown a)\smile(a\frown x)=(a\frown b)$ $\smile(a\frown x)\leqq a\frown(b\smile x)\leqq(a\smile b)\frown(b\smile x)=b$, this contradicts the hypothesis.

② If cdx, then $d=c\frown(d\smile x)\geqq d\smile(c\frown x)\geqq d$, and hence we have $d\smile(c\frown x)=d$, then $b\frown x\leqq c\frown x\leqq d$. However dbx does not hold, for $d\smile(b\frown x)=d\neq b$.

Corollary. *In a lattice quad-rangle abcd we have*

① $$ab//ba$$

② $$ca//ab$$

③ $$ab//cd.$$

Proof. ① $B^*(a,b)=B^*(a\smile b,b)\frown B^*(a\frown b,b)$, $B^*(b,a)=B^*(b\smile a,a)$ $\frown B^*(b\frown a,a)$ by (1), §5 [3]. Since $B^*(a\smile b,b)\frown B^*(a\frown b,a)=0$ from Theorem 2.1 we have $ab//ba$. ② By (1), §5 [3], we have $B^*(a,b)\subset$ $B^*(d,b)$ and $B^*(d,b)\frown B^*(c,a)=0$ by Theorem 2.1, hence we have $ca//ab$. The proof of ③ is obvious from ②, Theorem 2.1 and (1), §5 [3].

Theorem 2.2. *In a lattice quad-rangle abcd, if there exist two elements e,f such that* $a<e<c, d<f<b, b\frown e=f, a\smile f=e$, *then we have*

① $$be//af$$

② $$eb//fa.$$

Proof. ① At first we shall prove $ef//fe$. Suppose that $e>f$, efx, fex; then $f=(e\smile f)\frown(f\smile x)\geqq f\smile(e\frown x)=e$, this contradicts the hypothesis. On the other hand, $B^*(b,e)=B^*(c,e)\frown B^*(f,e)$, $B^*(a,f)$ $=B^*(e,f)\frown B^*(d,f)$ by (1), §5 [3]; thus we have $be//af$. ② is obtained from $cb//da$.

Theorem 2.3. *In a lattice polygon in L which consists of the two maximal chains* $\{a_n\}$, $\{b_m\}$ *with the condition* (C) *such that* $a_n\succ a_{n-1}\succ$ $\cdots\succ a_i\succ\cdots\succ a_0$, $b_m\succ b_{m-1}\succ\cdots\succ b_j\succ\cdots\succ b_0$, $a_0=b_0$, $a_n=b_m$,

(C) $$a_i\frown b_j=a_0,\ a_i\smile b_j=a_n.$$

We have $a_0a_i//b_lb_k$, *where* $i=1,2,\cdots,n$, $l=m, m-1,\cdots,2$; $l>k\geqq 1$.

Proof. If a_0a_ix, then $a_i=a_0\smile(a_i\frown x)$. However b_lb_kx does not hold, that is, $b_l\frown(b_k\smile x)\neq b_k$. Indeed we have $b_k\smile x\geqq b_k\smile a_0\smile(a_i\frown x)$ $=b_k\smile a_i=a_n$. Hence $b_l\frown(b_k\smile x)=b_l\neq b_k$.

Corollary. *In the lattice polygon of* Theorem 2.3, *we have*

① $a_ib_k//a_ha_i$, $a_ib_k//a_ka_i$, $i=1,2,\cdots,n$, $k=0,1,\cdots,m$, $0\leqq h<i$, $i<k\leqq n$.

② $a_s a_{s'} // b_k a_i$, $a_t a_{t'} // b_k a_i$, $i=1,2,\cdots,n-1, k=1,2,\cdots,m-1$, $0 \leqq s' < s \leqq i$, $i \leqq t < t' \leqq n$.

Proof. ① Since $B^*(a_i, b_k) \subset B^*(a_n, b_k)$ by (1), §5 [3] we have $a_n b_k // a_h a_i$ by the dual of Theorem 2.3, and hence we have $a_i b_k // a_h a_i$. Similarly we have the proof of ②.

Theorem 2.4. *In order that the lattice polygon of* Theorem 2.3 *has a diagonal, that is, it has the condition* (D) *instead of* (C), *it is necessary and sufficient that the condition* (E) *below holds.*

(D) $a_k > b_l$, $a_{k'} \smile b_{l'} = a_n \cdot a_{k'} \frown b_{l'} = b_l$ *for* $k \leqq k' < n$, $l < l' \leqq m$ *and* $a_{k''} \frown b_{l''} = a_0$, $a_{k''} \smile b_{l''} = a_k$ *for* $0 < k'' \leqq k$, $0 < l'' < l$.

(E) $a_k b_l \not\parallel b_l b_{l'}$, $l \neq 0$, m; $k \neq 0, n$.

Proof. If the lattice polygon has the condition (D), then we have $a_k b_l b_{l'}$, hence $B^*(a_k, b_l) \frown B^*(b_l, b_{l'}) \ni b_{l'}$, that is, $a_k b_l \not\parallel b_l b_{l'}$. If it has no diagonal, then we have $a_k b_l // b_l b_{l'}$ by the corollary of Theorem 2.3.

Corollary. *In the lattice polygon with the condition* (D), *we have*

① $$a_{k'} b_{l'} // b_{l''} a_{k''},$$

② $$B^*(a_{k''}, b_{l'}) \subset B^*(a_{k'}, b_{l'}).$$

Proof. ① By (1), §5 [3], we have $B^*(a_{k'}, b_{l'}) \subset B^*(a_n, b_{l'})$, $B^*(b_{l''}, a_{k''}) \subset B^*(a_0, a_{k''})$. However $a_0 a_{k''} // a_n b_{l'}$ by Theorem 2.3, and hence we have $a_{k'} b_{l'} // b_{l''} a_{k''}$.

② Since $a_0 b_l b_{l'}$, $a_0 b_{l'} x$ imply $b_l b_{l'} x$ by Lemma 4 [2], we have $B^*(a_0, b_{l'}) \subset B^*(b_l, b_{l'})$. Hence we have $B^*(a_{k''}, b_{l'}) \subset B^*(a_{k'}, b_{l'})$ by (1), §5 [3].

(a,b) $\overline{M}^*$ means that a,b do not form a relative modular pair (cf. [4]).

Theorem 2.5. *The necessary and sufficient condition for* (a,b) $\overline{M}^*$ *in a lattice quad-rangle abcd is that there exists at least one element* b' *such that* $ab' // b'b$, $d < b' < b$.

Proof. If (a,b) $\overline{M}^*$, there exists b' such that $b' \smile (a \frown b) = b' < (b' \smile a) \frown b$, $d < b' < b$. Let $b' \smile a = f$, $f \frown b = b''$, then we have $f \leqq c$, $d < b' < b'' \leqq b$. In this case if $b'bx$, then we have $b' \smile (b \frown x) = b$. However $fb'x$ does not hold, for $(f \smile b') \frown (b' \smile x) = f \frown (b' \smile x) \geqq f \frown (b' \smile (b \frown x)) = f \frown b = b'' > b'$. Hence we have $ab' // b'b$ since $B^*(a,b') = B^*(f,b') \frown B^*(d,b')$.

Next if (a,b) M^*, then for any $d < b' < b$ we have $b' \in B(a,b)$, that is, there exists c' such that $c' > b'$, $c > c' > a$. Then we have $ab'b$, and hence $b \in B^*(a,b')$, thus $B^*(a,b') \frown B^*(b',b) \neq 0$, this completes the proof.

Following L. R. Wilcox [5] by $a // b$ we mean that $a \frown b = 0$, (a,b) $\overline{M}^*$.

Corollary. *The necessary and sufficient condition for* $a // b$ *in a lattice quad-rangle abcd, where* $d=0$, *is* $ab' // b'b$ *for* $d < b' < b$.

References

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