93. A Generalization of Morita's Theorem concerning Generalized Uni-serial Algebras

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Recently K. Morita [1] has obtained the following theorem: a finite dimensional (associative) algebra A over a commutative field is generalized uni-serial if and only if every residue class algebra of A is a QF-3 algebra.

In this note we shall establish, by making use of notion of QF-3 rings introduced by H. Tachikawa, a generalization of this theorem for the case of rings with minimum condition for left and right ideals.

1. Throughout this note a ring will be assumed to have a unit element 1 and to satisfy the minimum (whence the maximum) condition for left and right ideals.

Following R. M. Thrall [2] we shall say that for a ring A, a faithful left A-module U is a minimal faithful left A-module if deletion of any direct summand of U leaves non-faithful left A-module. A ring A is said to be a QF-3 ring if it has a unique minimal faithful left A-module.

H. Tachikawa [3] has shown that A is a QF-3 ring if and only if there exists a faithful, projective, injective, left A-module; if A is a QF-3 ring, the direct sum of U_λ ($\lambda=1,\cdots,m$) is the unique minimal faithful left A-module where $U_1,\cdots,U_m$ are a representative set of primitive left ideals which are injective (and projective).

Now our theorem is stated as follows.

Theorem. *A ring A is generalized uni-serial if and only if every residue class ring of A is a* QF-3 *ring.*

Proof. Suppose that A is generalized uni-serial. Then every residue class ring of A is generalized uni-serial and hence it is a QF-3 ring. This proves the "only if" part.

Conversely, suppose that every residue class ring of A is a QF-3 ring. It is sufficient, by a theorem of T. Nakayama [4], to show that the residue class ring A/N^2 is generalized uni-serial, where N is the radical of A. Hence we have only to prove the following lemma.

Lemma. *Let A be a* QF-3 *ring such that $N^2=0$ where N is the radical of A. Then A is generalized uni-serial.*

2. *Proof of the lemma.* As is well known, we have a direct-sum decomposition of A into indecomposable left [right] ideals:

$$A=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}A,$$

where $e_{\kappa,i}$ ($\kappa=1,\cdots,n$; $i=1,\cdots,f(\kappa)$) are mutually orthogonal idempotents such that $Ae_{\kappa,i}\cong Ae_{\kappa,1}=Ae_{\kappa}$ for $i=1,\cdots,f(\kappa)$, and $Ae_{\kappa,i}\not\cong Ae_{\lambda,j}$ if $\kappa\neq\lambda$.

Since A is a QF-3 ring, A has a unique minimal faithful, projective, injective, left A-module $U=\sum_{\lambda\in\Pi}Ae_{\lambda}$, where Ae_{λ} ranges over all indecomposable, injective, left ideals in A. Here, by a theorem of [5], $Ae_{\lambda}(\lambda\in\Pi)$ has a unique simple subideal; $S(Ae_{\lambda})$*) $\cong Ae_{\varphi(\lambda)}/Ne_{\varphi(\lambda)}$, and $\varphi(\lambda)=\varphi(\kappa)$ if and only if $\lambda=\kappa$ for $\lambda,\kappa\in\Pi$.

Furthermore we want to show that if $\mu\notin\Pi$ then Ae_{μ} is a simple left ideal. For any non-zero element xe_{μ} of Ae_{μ}, there exists an element $\sum_{\lambda\in\Pi}a_{\lambda}e_{\lambda}$ in U such that $xe_{\mu}(\sum a_{\lambda}e_{\lambda})\neq 0$ because U is faithful. Hence, from the assumption that $N^2=0$ and the fact that $e_{\mu}(\sum a_{\lambda}e_{\lambda})\in N$, it follows that $xe_{\mu}\notin N$, and so Ae_{μ} is a simple left ideal. Consequently we see that A is a left generalized uni-serial ring.

Next we shall show that A is a right generalized uni-serial ring. To prove this, assume that $e_{\nu}N(\neq 0)$ is decomposed into simple constituents:

$$e_{\nu}N=e_{\nu}a_1e_{\kappa_1}A\oplus e_{\nu}a_2e_{\kappa_2}A\oplus\cdots\oplus e_{\nu}a_te_{\kappa_t}A\quad(t\geqq 2).$$

In this case, since $Ne_{\kappa_i}\cong Ae_{\nu}/Ne_{\nu}$ for all i, we have $\kappa_1=\kappa_2=\cdots=\kappa_t$ $(=\varphi^{-1}(\nu))$. We put $\kappa_1=\kappa$.

We now construct the interlacing module $\mathfrak{M}=Ae_{\kappa}\alpha_1\smile Ae_{\kappa}\alpha_2$ with the property that $xe_{\kappa}\alpha_i=0\Longleftrightarrow xe_{\kappa}=0$ $(i=1,2)$ and $xe_{\nu}a_1e_{\kappa}\alpha_1=xe_{\nu}a_2e_{\kappa}\alpha_2$ for every $x\in A$. We shall prove that $S(\mathfrak{M})\cong Ae_{\nu}/Ne_{\nu}$.

It is obvious that $N\mathfrak{M}$ is contained in $S(\mathfrak{M})$. Conversely let $Ae_{\mu}\xi$ be a simple submodule of $\mathfrak{M}$ which is not contained in $N\mathfrak{M}$. Evidently $\mu=\kappa$ and $e_{\kappa}\xi$ is expressed in the form:

$$e_{\kappa}\xi=e_{\kappa}xe_{\kappa}\alpha_1+e_{\kappa}ye_{\kappa}\alpha_2.$$

Then we have either $e_{\kappa}xe_{\kappa}\notin N$ or $e_{\kappa}ye_{\kappa}\notin N$; otherwise we would have $e_{\kappa}\xi\in N\mathfrak{M}$. Hence we may assume, without loss of generality, that $e_{\kappa}xe_{\kappa}\notin N$. Since $Ne_{\kappa}\xi=0$, we have $Ne_{\kappa}(e_{\kappa}xe_{\kappa})^{-1}e_{\kappa}\xi=0$; since $e_{\nu}a_1e_{\kappa}\in e_{\nu}N\subseteqq N$ we have

$$e_{\nu}a_1e_{\kappa}\alpha_1=-e_{\nu}a_1e_{\kappa}(e_{\kappa}xe_{\kappa})^{-1}e_{\kappa}ye_{\kappa}\alpha_2.$$

On the other hand, from the construction of $\mathfrak{M}$ we see that $e_{\nu}a_1e_{\kappa}\alpha_1=e_{\nu}a_2e_{\kappa}\alpha_2$. Thus we have $e_{\nu}a_2e_{\kappa}\alpha_2=-e_{\nu}a_1e_{\kappa}(e_{\kappa}xe_{\kappa})^{-1}e_{\kappa}ye_{\kappa}\alpha_2$ and hence $e_{\nu}a_2e_{\kappa}=-e_{\nu}a_1e_{\kappa}(e_{\kappa}xe_{\kappa})^{-1}e_{\kappa}ye_{\kappa}$, which contradicts the property that $e_{\nu}a_1e_{\kappa}A\frown e_{\nu}a_2e_{\kappa}A=0$. Therefore $S(\mathfrak{M})=N\mathfrak{M}$ $(=Ne_{\kappa}\alpha_1=Ne_{\kappa}\alpha_2)$, which is isomorphic to Ae_{ν}/Ne_{ν}.

Now Ae_{κ} is not simple since Ne_{κ} contains a non-zero element $e_{\nu}a_1e_{\kappa}$.

*) For a left A-module M, we denote the semi-simple part of M by $S(M)$; $S(M)=\{x\mid Nx=0,\ x\in M\}$.

As was proved at the beginning of the proof, a primitive left ideal which is not injective is simple. Therefore Ae_κ is an injective module with $S(Ae_\kappa) \cong Ae_\nu/Ne_\nu$. Hence there exists a monomorphism $\psi : \mathfrak{M} \rightarrow Ae_\kappa$, by a theorem of [5]. But the composition length of $\mathfrak{M}$ is three and that of Ae_κ is two since Ae_κ is injective. This contradicts our situation. Consequently, A is right generalized uni-serial and hence generalized uni-serial. Thus the lemma is completely proved.

References

[1] K. Morita: Duality for modules and its applications to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku, **6**, no. 150, 83–142 (1958).
[2] R. M. Thrall: Some generalizations of quasi-Frobenius algebras, Trans. Amer. Math. Soc., **64**, 173–183 (1948).
[3] H. Tachikawa: Note on QF-3 rings (to appear).
[4] T. Nakayama: Note on uni-serial and generalized uni-serial rings, Proc. Imp. Acad., **16**, 285–289 (1940).
[5] K. Morita, Y. Kawada, and H. Tachikawa: On injective modules, Math. Zeitschr., **68**, 217–226 (1957).