120. A Generalization of Vainberg's Theorem. I

By Tetsuya SHIMOGAKI
Mathematical Institute, Hokkaidô University, Sapporo
(Comm. by K. KUNUGI, M.J.A., Oct. 13, 1958)

1. Let E be a measurable set in Euclidean n-space and $f(u,t)$ be a real valued function defined for u real and t in E such that it is *continuous* as a function of u for almost all $t \in E$ and *measurable* as a function of t for all u.

By this function $f(u,t)$ we define for every real valued measurable function $x(t)$

(1.1) $$\mathfrak{H}(x(t)) = f(x(t), t).$$

Then $\mathfrak{H}(x(t))$ is also measurable function on E and $\mathfrak{H}$ establishes a transformation on the space of measurable functions on E into itself.

Recently in [2] M. M. Vainberg proved that *in order that* $\mathfrak{H} \equiv f(u,t)$ *maps* $L_p(E)$ *into* $L_{p_1}(E)$ $(p, p_1 > 0)$ *it is necessary and sufficient that there exist a positive number* γ *and a function* $a(t)$ *belonging to* $L_{p_1}(E)$ *such that*

(1.2) $$|f(u,t)| \leq a(t) + \gamma |u|^{\frac{p}{p_1}}$$

for all $t \in E$, $u \in (-\infty, +\infty)$.

Let B be a Banach space consisting of measurable functions on E and B^* is its *conjugate* space. The operators $\mathfrak{H} \equiv f(u,t)$ which map B into B^* are particularly interesting and discussed by several authors, because of their connection to the theory of non-linear integral operators of the form:

(1.3) $$Ax(t) = \int_E K(t,s) f(x(s), s) ds.$$

We shall generalize the Vainberg's Theorem on *modulared semi-ordered linear spaces* and point out that $\mathfrak{H}$ is characterized by *conjugately similar correspondences*[1)] [1, § 59], in the case that $\mathfrak{H}$ operates into the conjugate spaces. Here we shall prove only the fundamental theorem, which allows to obtain the Vainberg's Theorem in more general form. For want of space the details will be discussed in the following paper.

2. Let R be a *modulared semi-ordered linear space*,[2)] and $m(a)$ $(a \in R)$ be *a modular* on R. The totality of all elements $a \in R$ such that

1) The definition of the conjugately similar correspondence will be stated in the following paper.

2) We suppose that semi-ordered linear space is always universally continuous in the sequel, i.e. $a_\lambda \geq 0$ $(\lambda \in \Lambda)$ implies $\bigcap_{\lambda \in \Lambda} a_\lambda \in R$. The notations and terminologies used here are the same ones used in [1].

$m(\xi a)<+\infty$ for every $\xi\geq 0$ is called the *finite manifold of R by m* and denoted by F_m. If F_m is *complete* (i.e. $|a|\frown|b|=0$ for all $b\in F_m$ implies $a=0$), then the modular m is said to be *almost finite*. If $F_m=R$, m is said to be *finite*. From the definition we can see easily that *if a modular m is almost finite then for every $x\in R$, there exist $[p_\lambda]\uparrow_{\lambda\in\Lambda}[x]$ such that $m([p_\lambda]x)<+\infty$.* A modular m is said to be *monotone complete*, if $0\leq a_\lambda\uparrow_{\lambda\in\Lambda}$, $\sup_{\lambda\in\Lambda} m(a_\lambda)<+\infty$ implies $\bigcup_{\lambda\in\Lambda} a_\lambda\in R$. $L_p(E)$ spaces ($p\geq 1$) and Orlicz spaces $L_\Phi(E)$ are examples of monotone complete modulared spaces, with modulars

$$m_p(a)=\int_E |a(t)|^p dt,\qquad m_\Phi(b)=\int_E \Phi(|b(t)|)dt\ ^{3)}$$

for $a(t)\in L_p(E)$, $b(t)\in L_\Phi(E)$ respectively.

Definition. An operator H defined on R into itself is called to be *splitable* if it satisfies

(2.1) $$[N](Hx)=H([N]x)$$

for all $x\in R$ and $N\subset R$.[4]

Lemma 1. *If an operator H is splitable, then we have*

(2.2) $H(x+y)=Hx+Hy$ and $|Hx|\frown|Hy|=0$, *for $x, y\in R$ such that* $|x|\frown|y|=0$;

(2.3) *for every $x, y\in R$ there exists $z\in R$ such that $Hz=Hx\smile Hy$.*

Proof. For x, $y\in R$ such that $|x|\frown|y|=0$, we have $H(x+y)=H([x+y](x+y))=[x+y]H(x+y)=[x]H(x+y)+[y]H(x+y)=Hx+Hy$, since $[x+y]=[x]+[y]$ and $[x]y=[y]x=0$. Therefore (2.2) holds.

For every x, $y\in R$, $Hx\smile Hy=(Hx-Hy)^{+}$ [5] $+Hy=[(Hx-Hy)^+](Hx-Hy)+Hy$. Putting $c=(Hx-Hy)^+$ we have $Hx\smile Hy=[c]Hx+(1-[c])Hy=H([c]x)+H((1-[c])y)=H([c]x+(1-[c])y)$. Thus (2.3) holds with $z=[c]x+(1-[c])y$.

Lemma 2. *Let R be a modulared semi-ordered linear space whose modular m is monotone complete. And let ρ_ν ($\nu=1,2,\cdots$) be a sequence of functionals on R such that*

(2.4) $0\leq\rho_\nu(a)\leq+\infty$ *for every $a\in R$ and $\nu\geq 1$;*

(2.5) $\rho_\nu(a+b)\leq\rho_\nu(a)+\rho_\nu(b)$ *for every $|a|\frown|b|=0$ and $\nu\geq 1$;*

(2.6) $\sup_{\lambda\in\Lambda}\rho_\nu([p_\lambda]a)=\rho_\nu(a)$ *for every $[p_\lambda]\uparrow_{\lambda\in\Lambda}[a]$ and $\nu\geq 1$;*

(2.7) $\overline{\lim}_{\nu\to\infty}\rho_\nu(a)<+\infty$ *for every $a\in R$.*

Then there exist positive numbers ε,γ, a finite dimensional normal manifold N and a natural number ν_0 such that $m(x)\leq\varepsilon$, $x\in(1-[N])R$ implies $\rho_\nu(x)\leq\gamma$ for every $\nu\geq\nu_0$.

3) In Orlicz space $L_\Phi(E)$, m_Φ is almost finite if and only if $\Phi(u)<+\infty$ for all $0\leq u<+\infty$.

4) $[N]$ is a projection operator defined by the least normal manifold including N.

5) $a^+(a\in R)$ means the positive part of a.

Proof. The set $\mathfrak{E}$ of all maximal ideals $\mathfrak{p}$[6] of normal manifolds constitutes a compact Hausdorff space with a neighbourhood system: $\{U_{[N]}: N\in\mathfrak{p}\}$ where $U_{[N]}$ is a set of all maximal ideals $\mathfrak{p}$ to which N belongs [1, §8]. We shall first show that *for arbitrary non-atomic maximal ideal*[7] $\mathfrak{p}$ *of normal manifolds, we can find a normal manifold* $N_{\mathfrak{p}}\in\mathfrak{p}$, *positive numbers* $\varepsilon_{\mathfrak{p}}, \gamma_{\mathfrak{p}}$ *and a natural number* $\nu_{\mathfrak{p}}$ *such that*

$$\sup_{m(x)\leq\varepsilon, x\in N_{\mathfrak{p}}} \rho_\nu(x) \leq \gamma_{\mathfrak{p}} \qquad \textit{for every } \nu\geq\nu_{\mathfrak{p}}.$$

We suppose that this statement is not valid for a non-atomic maximal ideal $\mathfrak{p}_0$ of normal manifolds. Then we shall construct a sequence of orthogonal elements $x_1, x_2,\cdots, x_\nu,\cdots$ such that $[x_1, x_2,\cdots, x_\nu]R\bar{\in}\mathfrak{p}_0$, $m(x_\nu)\leq\frac{1}{2^\nu}$, $\rho_{\kappa(\nu)}(x_\nu)\geq\nu$ and $\kappa(\nu+1)>\kappa(\nu)$ for all $\nu\geq 1$.

In fact, let us assume that $x_1, x_2,\cdots, x_\nu$ have been chosen as above. Since $\mathfrak{p}_0$ is maximal, $(1-[x_1, x_2,\cdots, x_\nu])R\in\mathfrak{p}_0$. Now we can find an element $x\in(1-[x_1, x_2,\cdots, x_\nu])R$ such that $m(x)\leq\frac{1}{2^{\nu+1}}$, $\rho_{\kappa(\nu+1)}(x)>\nu+1$ and $\kappa(\nu+1)>\kappa(\nu)$ by the assumption. Since $\mathfrak{p}_0$ is non-atomic and ρ_ν satisfy the condition (2.6) there exists a normal manifold $N\bar{\in}\mathfrak{p}_0$ such that $\rho_{\kappa(\nu+1)}([N]x)\geq\nu+1$. Here we can put $x_{\nu+1}=[N]x$ because $x_{\nu+1}\in(1-[x_1, x_2,\cdots, x_\nu])R$, $[x_1, x_2,\cdots, x_\nu, x_{\nu+1}]R\bar{\in}\mathfrak{p}_0$, $m(x_{\nu+1})\leq\frac{1}{2^{\nu+1}}$ and $\rho_{\kappa(\nu+1)}(x)\geq\nu+1$.

For the sequence thus obtained, since $m(\bigcup_{\nu=1}^{\mu} x_\nu)\leq 1$ for every $\mu\geq 1$, $x_0=\bigcup_{\nu=1}^{\infty} x$ exists in R because of monotone completeness of m. On the other hand, since $\rho_{\kappa(\nu)}(x_0)\geq\rho_{\kappa(\nu)}([x_\nu]x_0)=\rho_{\kappa(\nu)}(x_\nu)$, we obtain

$$\overline{\lim_{\nu\to\infty}}\,\rho_\nu(x_0)=\overline{\lim_{\nu\to\infty}}\,\rho_{\kappa(\nu)}([x_\nu]x_0)=\lim_{\nu\to\infty}\rho_{\kappa(\nu)}(x_\nu)=+\infty,$$

which contradicts (2.7).

Therefore we have shown that for every non-atomic $\mathfrak{p}$ there exist a normal manifold $N_{\mathfrak{p}}$, positive numbers $\varepsilon_{\mathfrak{p}}, \gamma_{\mathfrak{p}}$ and a natural number $\nu_{\mathfrak{p}}$ for which the above statement holds.

We denote by $\mathfrak{a}$ the totality of all non-atomic $\mathfrak{p}\in\mathfrak{E}$. And let N_0 be the least normal manifold including all $N_{\mathfrak{p}}$ $(\mathfrak{p}\in\mathfrak{a})$. Then we have $U_{[N_0]}=(\sum_{\mathfrak{p}\in\mathfrak{a}}^{k} U_{[N_{\mathfrak{p}}]})^-$ [1, §8]. On the other hand each $\mathfrak{p}\in U_{[N_0]}-\sum_{\mathfrak{p}\in\mathfrak{a}}^{k} U_{[N_{\mathfrak{p}}]}$ is non-atomic as easily seen, we have $U_{[N_0]}=\sum_{\mathfrak{p}\in\mathfrak{a}} U_{[N_{\mathfrak{p}}]}$. Since $U_{[N_0]}$ is compact, there exists a finite number of $\mathfrak{p}\in\mathfrak{a}$, that is, $\mathfrak{p}_1, \mathfrak{p}_2,\cdots, \mathfrak{p}_\kappa$ such that $U_{[N_0]}\subseteq\sum_{\nu=1}^{k} U_{[N_{\mathfrak{p}_\nu}]}$. Putting $\gamma=\sum_{\nu=1}^{k}\gamma_{\mathfrak{p}_\nu}$, $\varepsilon=\min_{\nu=1,2,\cdots,\kappa}\{\varepsilon_{\mathfrak{p}_\nu}\}$ and $\mu_0=\max_{\nu=1,2,\cdots,\kappa}\{\nu_{\mathfrak{p}_\nu}\}$

6) A system $\mathfrak{p}$ of normal manifolds is called an ideal if i) $\mathfrak{p}\bar{\ni}0$; ii) $\mathfrak{p}\ni M\subset N$ implies $\mathfrak{p}\ni N$; iii) $\mathfrak{p}\ni N, M$ implies $\mathfrak{p}\ni N\cdot M$. An ideal $\mathfrak{p}$ is said to be maximal if there exists no other ideal containing $\mathfrak{p}$ [1, §8].

7) An ideal $\mathfrak{p}$ is said to be *non-atomic*, if $\mathfrak{p}\ni N$, there exists $M\subsetneqq N$ such that $M\in\mathfrak{p}$.

we obtain by (2.5) and (2.6)

$$\rho_\mu(x) \leq \sum_{\nu=1}^{\kappa} \rho_\mu([N_{\mathfrak{p}_\nu}]x) \leq \sum_{\nu=1}^{\kappa} \gamma_{\mathfrak{p}_\nu} = \gamma$$

for any $x \in [N_0]R$, $m(x) \leq \varepsilon$ and $\mu \geq \mu_0$.

Here $(1-[N_0])R$ is finite-dimensional, because, when $(1-[N_0])R$ is infinite-dimensional, there exists at least a non-atomic maximal ideal $\mathfrak{p} \in \mathfrak{E}$ such that $\mathfrak{p} \ni (1-[N_0])R$. Thus Lemma is proved.

R is said to be *non-atomic* if every $x \in R$ with $x \neq 0$ can be decomposed into two orthogonal elements: $x=y+z$, $y, z \neq 0$, $|y| \frown |z|=0$.

Now we shall prove the following fundamental theorem:

Theorem 1. *Let R be a non-atomic modulared semi-ordered linear space whose modular m is monotone complete. Then in order that for any splitable operator H on R there exist a positive number $\gamma>0$ and an element $R \ni c \geq 0$ such that*

(2.8) $$|Hx| \leq c+\gamma|x| \qquad \textit{for all } x \in R,$$

it is necessary and sufficient that m is almost finite.

Proof. Sufficiency. For a splitable operator H we put $\rho_\nu(x) = m\left(\frac{1}{\nu} Hx\right)$ for $\nu \geq 1$ and $x \in R$. Since for every $[p_\lambda] \uparrow_{\lambda \in \Lambda} [x]$

$$\bigcup_{\lambda \in \Lambda} H([p_\lambda]x) = \bigcup_{\lambda \in \Lambda} [p_\lambda]Hx = [x]Hx = Hx,$$

$\rho_\nu(x)$ satisfies the condition (2.6) for each $\nu \geq 1$ by virtue of semi-continuity of the modular m. Because of the modular conditions and formula (2.2), we can see that the functionals ρ_ν $(\nu \geq 1)$ satisfy (2.4), (2.5) and (2.7) in the previous lemma. Hence there exist positive number ε, δ and a natural number μ_0 such that $m(x) \leq \varepsilon$ implies $m\left(\frac{1}{\mu_0} Hx\right) \leq \delta$. Let γ be a positive number such that $\frac{\gamma}{\mu_0}\varepsilon > \gamma$ and $\frac{\gamma}{\mu_0} > 1$.

We define an operator T for $x \in R$ as

$$Tx = |Hx| - \gamma|x|.$$

Since $T([N]x) = |H([N]x)| - \gamma|[N]x| = [N](|Hx| - \gamma|x|)$, T is also splitable. Suppose that $Tx \geq 0$ and $m(x) > \varepsilon$. Then we can find a projection operator $[N]$ such that $m([N]x) = \varepsilon$, since m is almost finite and R is non-atomic. For such $[N]x$ we have

$$\delta \geq m\left(\frac{1}{\mu_0} H([N]x)\right) \geq m\left([N]\frac{1}{\mu_0}|\gamma x|\right) \geq \frac{\gamma}{\mu_0} m([N]x) > \delta,$$

which is a contradiction. Therefore $Tx \geq 0$ implies $m(x) \leq \varepsilon$ and fortiori $m\left(\frac{1}{\mu_0} Tx\right) = m\left(\frac{1}{\mu_0}(|Hx| - \gamma|x|)^+\right) \leq m\left(\frac{1}{\mu_0}|Hx|\right) \leq \delta.$

By virtue of (2.3), the set $\{Tx : x \in R\}$ is directed, so there exists $0 \leq \bigcup_{x \in R} Tx \in R$, because of monotone completeness of m. Thus we have

$$|Hx| \leq \bigcup_{x \in R} Tx + \gamma|x| \qquad \text{for all } x \in R.\text{[8]}$$

8) The author indebted to Prof. I. Amemiya for this proof, simpler than the original one.

Necessity. If m is not almost finite, then we can find an element $R\ni a_0\geq 0$ such that [1, §35, §45]

$$m([p]a_0)=+\infty \qquad \text{if } [p][a_0]\neq 0.$$

a_0 is a strong unit in $[a_0]R$, i.e. for every $x\in[a_0]R$ there exists $\xi\geq 0$ such that $|x|\leq\xi a_0$.

By assumption R is non-atomic, then we can decompose a_0 into infinite numbers of positive orthogonal elements: $a_0=\sum_{\nu=1}^{\infty}a_\nu$, $a_\nu\frown a_\mu=0$ $(\nu\neq\mu)$. For every ν we define an operator $\mathfrak{h}_\nu$ on $[a_\nu]R$ as

$$\mathfrak{h}_\nu x=\int_{[a_\nu]}|\varphi_\nu(x,\mathfrak{p})|\,d\mathfrak{p}\nu a \qquad (x\in[a_\nu]R),$$

where
$$\varphi_\nu(x,\mathfrak{p})=\begin{cases}\left(\dfrac{x}{\nu a_\nu},\mathfrak{p}\right)^{9)} & \text{if } \left(\dfrac{x}{\nu a_\nu},\mathfrak{p}\right)\leq 1,\\ \left(\dfrac{x}{\nu a_\nu},\mathfrak{p}\right)^2 & \text{if } \left(\dfrac{x}{\nu a_\nu},\mathfrak{p}\right)>1.\end{cases}$$

For any $x\in R$, putting $Hx=\sum_{\nu=1}^{\infty}\mathfrak{h}_\nu[a_\nu]x$ we obtain a splitable operator on R into itself. Since $H(\nu^2a_\nu)=\nu^3a_\nu$, H has not the form (2.8) in any way. The proof is completed.

When R is a *discrete modulared space,* Theorem 1 does not remain true. Here we shall comment shortly on the case that R is discrete.

Let R_d be an *almost finite, monotone complete modulared space which is discrete.* Then there exist $R\ni e_\lambda\geq 0$ $(\lambda\in\Lambda)$ such that $m(e_\lambda)=1$, $e_\gamma\frown e_\lambda=0$ if $\lambda\neq\gamma$, and for any $0\leq x\in R$ we can find the positive numbers $\xi_\lambda=\xi_\lambda(x)\geq 0$ $(\lambda\in\Lambda)$ for which $x=\bigcup_{\lambda\in\Lambda}\xi_\lambda e_\lambda$ holds. For arbitrary positive number α we denote by B_α the totality of all $x\in R$ such that $|\xi_\lambda(x)|\leq\alpha$ for all $\lambda\in\Lambda$. When R is discrete, the theorem corresponding to the previous one is stated as follows:

Theorem 1′. *For any splitable operator H on R_d into itself there exist positive numbers γ, α, a finite dimensional normal manifold M and an element $c\in R_d$ for which formula* (2.8) *holds for every* $x\in(1-[M])B_\alpha$.

The method of proof of this theorem is analogous to those of Lemma 2 and Theorem 1, thus we omit it here.

Remark 1. When R is a non-atomic normed semi-ordered linear space, $\rho_\nu(x)=\|x\|$ $(x\in R)$ satisfy the conditions (2.4), (2.5) and (2.7). Thus from the proofs of Lemma 2 and Theorem 1 we can conclude that any splitable operator H on R into itself has the form (2.8) if the norm $\|x\|$ on R satisfies i) $\|x\|$ is monotone complete, i.e. $0\leq x_{\lambda\in\Lambda}\uparrow$,

9) $\left(\dfrac{b}{a},\mathfrak{p}\right)$ is a relative spectrum of b by a at $\mathfrak{p}$ [1, §10].

$\sup_{\lambda \in \Lambda} \|x_\lambda\| < +\infty$ implies $\bigcup_{\lambda \in \Lambda} x \in R$; ii) for any $x \in R$, and ε $(0 < \varepsilon < \|x\|)$ there exists $[p]$ such that $\|[p]x\| = \varepsilon$.

Remark 2. Let R be a semi-ordered linear space satisfying the all hypotheses of Theorem 1 (or Remark 1) and C be a subset of R with the conditions: i) $a \in C$, $|b| \leq |a|$ implies $b \in C$; ii) $a, b \in C$, $|a| \cap |b| = 0$ implies $a + b \in C$; iii) $0 \leq a_\lambda (\lambda \in \Lambda)$, $\sup_{\lambda \in \Lambda} m(a_\lambda) < +\infty$ implies $\bigcup_{\lambda \in \Lambda} a_\lambda \in C$. Then a splitable operator defined on C into R has the form (2.8) for all $x \in C$. This is ascertained by the proofs of Lemmas and Theorem 1.

References

[1] H. Nakano: Modulared Semi-ordered Linear Spaces, Tokyo (1950).

[2] M. M. Vainberg: On a structure of an operator, Doklady Acad. Nauk (n.s.), **92**, no. 2 (1953).