15. Some Properties of F-spaces

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X[1] is called an *F-space* provided for any $f\in C(X)$, $P(f)=\{x; f(x)>0\}$ and $N(f)=\{x; f(x)<0\}$ are completely separated. X has the *F_σ-property* if the closure of any F_σ-open subset of X is open. X has the *E_σ-property* if any $f\in B(U)$ has a continuous extension over X where U is any F_σ-open subset of X. Gillman and Henriksen [1] have proved the interest results on F-spaces; for instance, i) X is σ-complete if and only if for any $f\in C(X)$, $\overline{P(f)}$ is open; ii) X is an F-space if and only if any $f\in B(X-N)$ has a continuous extension over X where N is any Z-set of X. In general, 1) if X has the F_σ-property, X is σ-complete [3] and 2) if X has the E_σ-property, X is an F-space. If X is normal the converses of the above two statements are true [3].

In § 1 we shall study the relations between a given space X and its Čech compactification $(=\beta X)$ concerning the F_σ-prop., E_σ-prop., σ-completeness, or the property of being an F-space. In § 2 we shall consider some questions arising in connection with the theorems in § 1.

1. Theorem 1. *The following conditions are equivalent for any space X: 1) X has the F_σ-property; 2) any subspace Y of βX containing X as a proper subset has the F_σ-property; 3) any proper F_σ-open subset of X has the F_σ-property.*

Proof. $(1\to 2)$. Let V be any F_σ-open subset of Y. $U=V\frown X$ is also F_σ-open in X and hence $\overline{U}(\text{in } X)$ is open in X. On the other hand, $\beta X=\beta(\overline{U}(\text{in } X))\smile\beta(X-\overline{U}(\text{in } X))$, $\beta(\overline{U}(\text{in } X))\frown\beta(X-\overline{U}(\text{in } X))=\theta$ and $\overline{U}(\text{in } \beta X)=\beta(\overline{U}(\text{in } X))$. Since X is dense in Y and $U=X\frown V$ and V is open in Y, we have $\overline{V}(\text{in } Y)=\overline{U}(\text{in } Y)=\overline{U}(\text{in } \beta X)\frown Y$ and hence $\overline{V}(\text{in } Y)$ is open.

$(2\to 3)$. Let U be a proper F_σ-open subset of X and let V be F_σ-open in U. V is F_σ-open in X and we put $Y=(\beta X-(\overline{V}(\text{in } \beta X)-V))\smile X$. Since V is F_σ-open in Y and Y has the F_σ-property, $\overline{V}(\text{in } Y)$ is open in Y and hence $\overline{V}(\text{in } U)=\overline{V}(\text{in } Y)\frown U$ is open in U.

$(3\to 1)$. Let U be any proper F_σ-open subset of X. Suppose that $\overline{U}\neq X$ and $a\in X-\overline{U}$. There exists $f\in B(X)$ such that $f(a)=0$ and

1) A space X considered here is always a completely regular T_1-space. The functions are assumed to be real-valued and $C(X)(B(X))$ denotes the totality of (bounded) continuous functions defined on X.

$f(x)=1$ on U. $P(f)$ is F_σ-open in X and $P(f)\supset U$. $\overline{U}$(in $P(f)$) is open in $P(f)$ by (3) and hence $\overline{U}$(in X)$=\overline{U}$(in $P(f)$) is open in X.

Theorem 2. *The following conditions are equivalent for any space X:* 1) *X has the E_σ-property;* 2) *any subspace Y of βX containing X as a proper subset has the E_σ-property;* 3) *any proper F_σ-open subset of X has the E_σ-property.*

Proof. ($1\rightarrow 2$). Let V be F_σ-open in Y and $g\in B(V)$. $U=X\frown V$ is F_σ-open in X. By the assumption, a function $f(=g|V)$ has a continuous extension h over X and hence over βX. $h|Y$ is an extension of g because U is dense in V.

($2\rightarrow 3$). Let U, V and Y be sets as in the proof ($2\rightarrow 3$) in Theorem 1 and $f\in B(V)$. Then by the assumption, f can be continuously extended over Y and hence over U.

($3\rightarrow 1$). Let U be an F_σ-open subset of X and $f\in B(U)$. We take a point p in U and an open neighborhood V of p contained in U. By the complete regularity of X, there exists $g\in B(X)$ such that $g\geq 0$, $g(p)=0$ and $g(x)=1$ on $X-V$. Then $P(g)$ is a proper F_σ-open subset of X and hence $P(g)\frown U$ is F_σ-open in $P(g)$. Therefore $f|(P(g)\frown U)$ has a continuous extension h over $P(g)$. Then a function $F(x)$ defined by $F(x)=h(x)$ for $x\in P(g)$ and $F(x)=f(x)$ for $x\in g^{-1}(0)$, is a continuous extension of f over X.

Theorem 3. *The following conditions are equivalent for any space X:* 1) *X is an F-space;* 2) *βX is an F-space;* 3) *$P(f)$ is an F-space for any $f\in C(X)$ such that $P(f)\neq X$.*

Proof. ($1\leftrightarrow 2$) is obtained by Gillman and Henriksen [1].

($1\rightarrow 3$). Suppose that $P(f)\neq X$ and $g\in B(P(f))$ and $M=P(f)-Z(g)$. We shall prove that any $h\in B(M)$ can be continuously extended over $P(f)$. Let $\varphi=f\vee 0$ on X. Then $P(f)=X-Z(\varphi)$ and hence g has a continuous extension g^* over X because $g\in B(X-Z(\varphi))$. Since $Z(\varphi g^*)=Z(\varphi)\smile Z(g^*)$, we have $h\in B(X-Z(\varphi g^*))$, therefore h has a continuous extension over X and hence over $P(f)$.

($3\rightarrow 1$). Let $f^*\in C(X)$ and $g\in B(X-Z(f^*))$. Since $Z(f)=Z(|f|)$, we assume that $f\geq 0$. For any (fixed) point $a\in P(f)$, there is $h\in B(X)$ such that $h(a)=0$ and $h(x)=1$ on $Z(f)$. $P(h)$ is an F-space and $g\in B(P(h)-Z(f'))$ where $f'=f|P(f)$, and hence g has a continuous extension g' on $P(h)$. Let us put $G(x)=g'(x)$ for $x\in P(h)$ and $G(x)=g(x)$ for $x\notin P(h)$. Then $G(x)$ is a continuous extension of g over X and $G(x)|P(f)=g$.

Theorem 4. *The following conditions are equivalent for any space X:* 1) *X is σ-complete;* 2) *βX is σ-complete;* 3) *$P(f)$ is σ-complete for any $f\in C(X)$ such that $P(f)\neq X$.*

Proof. ($1\leftrightarrow 2$). The arguments of this proof are essentially the same as those used in [7, Theorem 1].

($1\to 3$). Let $U=P(f)$, $f\in C(X)$. We shall prove that $\overline{P(g)}(\text{in } U)$ is open in U for every $g\in B(U)$. Since X is an F-space and $U=X-Z(f\vee 0)$, g has a continuous extension $\tilde{g}$ over X and $\overline{P(\tilde{g})}(\text{in } X)$ is open in X because X is σ-complete. On the other hand, since $\overline{U}(\text{in } X)$ is open, $\overline{P(g)}(\text{in } U)=\overline{P(g)}(\text{in } \overline{U})\frown U=\overline{P(g)}(\text{in } X)\frown U=\overline{P(\tilde{g})}(\text{in } X)\frown U$ and hence $\overline{P(g)}(\text{in } U)$ is open in U.

($3\to 1$). Let $f\in C(X)$, $f\geq 0$ and $U=P(f)$. If there is a point a in $X-U$, then there exists $g\in C(X)$ such that $g(a)=0$ and $g(x)=1$ on U. By the assumption, $P(g)$ is σ-complete and $P(f)\subset P(g)$ and hence $\overline{P(f)}(\text{in } P(g))$ is open in $P(g)$. Since $g(P(f))=1$, $\overline{P(f)}(\text{in } X)$ is open in X.

2. Let X be a space having the F_σ-property and Z any compactification of X. By Theorem 1, it is natural to consider the question whether the F_σ-property for any subspace of Z containing X as a proper dense subspace implies $\beta X=Z$ or not.[2)] In the following, we deal with this question and similar ones; we give negative answers for these problems. First we shall consider compact subsets in F-spaces.

Theorem 5.[3)] *If X is an F-space and A is any compact subset of X, then $A-A_1$ is countably compact where A_1 is any finite subset of A.*

Proof. It is sufficient to prove that $A-A_1$ is countably compact in case $A_1=\{x\}$; the general case will be treated similarly. Suppose that there exists a closed set $B\{x_n;\ n=1,2,\cdots\}$ in $A-A_1$ such that each point x_n is an isolated point in B. Since A is compact, we have $\overline{B}=B\smile\{x\}$. Let f be a function on $\overline{B}$ such that $f(x_{2n})=-1/2n$ and $f(x_{2n+1})=1/(2n+1)$ and $f(x)=0$. Since f is continuous on a compact subset $\overline{B}$, f has a continuous extension g over X. Then $P(g)$ and $N(g)$ are not completely separated. This contradicts the fact that X is an F-space.

Corollary 1. *In an F-space, there exist no compact subsets which are countable.*

Corollary 2. *If an F-space has a unique structure, then X is countably compact.*

From this Corollary 2, $X=[1,\Omega]\times[1,\omega]-\{(\Omega,\omega)\}$ is not an F-space because X has a unique structure but is not countably compact

2) In this question, if "F_σ-property" is replaced by "stoneanness" this question is affirmatively solved [7, Theorem 7]. For the special case, we have $\beta X=Z$ certainly because X is always the complement of Z-set of Z where X is a locally compact, σ-compact F-space.

3) This theorem is a generalization of Corollary 2.4 in [1] and Theorem 3 in [7].

where ω and Ω are first ordinals of the second and third classes.

It is obvious that if $X \supset Y \ni x$ and x is a P-point of X, then x is also a P-point of Y [2]. But the converse is not true in general.[4] If Y is a dense subset of X, then the converse is true; this is seen by the following

Lemma 1. *If Y is a dense subspace of X, then every P-point of Y is also a P-point of X.*

Proof. Let a be a P-point of Y. If a is an isolated point in Y, a is also an isolated point in X, and hence we can assume that a is not isolated. Suppose that $\{V_n;\ n=1,2,\cdots\}$ is a family of neighborhoods (in X) of a. By the regularity of X, there exists a family $\{U_n;\ n=1,2,\cdots\}$ of open sets containing a in X such that $V_n \supset \overline{U}_n$ and $U_n \supset \overline{U}_{n+1}$. Since a is a P-point of Y, there exists a neighborhood U in X of a such that $\bigcap_{n=1}^{\infty}(U_n \cap Y) \supset Y \frown U$. It is well known that $\overline{Y \frown U} \supset \overline{Y} \frown U = X \frown U = U$. Therefore we kave $\bigcap_{n=1}^{\infty} \overline{U}_n \supset U$ and hence $\bigcap_{n=1}^{\infty} V_n \supset U$. This shows that a is a P-point of X.

Lemma 2. *If U is an F-open subset of X and x is a P-point of X, then $U \not\ni x$ implies $\overline{U} \not\ni x$.*

In the following, M is a subspace of βX containing X. We denote by $Z=M(\{C\})$ a space which is obtained by contracting C to one point $c=p(C)$ where C is a closed subset in $M-X$, and $\varphi=\varphi(\{C\})$ denotes a closed continuous mapping of M onto Z such that $\varphi(x)=x$ for $x \notin C$ and $\varphi(x)=c$ for $x \in C$.

Theorem 6. *Suppose that C is a compact subset of $M-X$.* 1) *Let c be a P-point of Z; if X has the F_σ-property or is σ-complete respectively, then any subspace Y of Z containing X has the F_σ-property or is σ-complete respectively.* 2) *If C consists of P-points (and hence C is a finite set* [2]*), then the point c is a P-point of Z. In this case, if X is an F-space or has the E_σ-property respectively, then any subspace Y of Z containing X is an F-space or has the E_σ-property respectively.*

Proof. 1) Suppose that X has the F_σ-property, U is F_σ-open in Y. If $Y \not\ni c$, then Y can be regarded as a subspace of M and hence, by Theorem 1, $\varphi^{-1}(Y)$ has the F_σ-property. Since $\varphi | \varphi^{-1}(Y)$ is a homeomorphism, Y has the F_σ-property. If $Y \ni c$ and $U \not\ni c$, then $\overline{U}(\text{in } Y) \not\ni c$ by Lemma 2. Let $Q=\overline{\varphi^{-1}(U)}(\text{in } \varphi^{-1}(Y))$. Since $\varphi(Q) \subset \overline{U}(\text{in } Y)$, we have $Q \frown C = \theta$ and Q is open in $\varphi^{-1}(Y)$ by the assumption. Since $\varphi | (\varphi^{-1}(Y)-C)$ is a homeomorphism of $\varphi^{-1}(Y)-C$ onto $Y-c$, $\varphi(Q)$ is

4) Let N be the set of all natural numbers. $\beta N - N$ contains P-points, under the continum hypothesis [6], but it is easily seen that every point in $\beta N - N$ is not a P-point of βN.

open and closed, and we have $\varphi(Q)=\overline{U}(\text{in } Y)$, i.e. $\overline{U}(\text{in } Y)$ is open. If $Y \ni c$ and $U \ni c$, then the openness of $\overline{U}(\text{in } Y)$ is obvious. In case X is σ-complete, our assertion will be obtained replacing U by $P(f)$ for any $f \in C(Y)$.

2) Suppose that c is not a P-point of Z. Let U be any F_σ-open subset of Z such that $\overline{U}(\text{in } Z) \ni c$ and $U \not\ni c$. $V=\varphi^{-1}(U)$ is also F_σ-open in M and hence we have $\overline{V}(\text{in } M) \frown C=\theta$ by Lemma 2. Since φ is a closed mapping and $\varphi(C)=c$, we have $\varphi(\overline{V}(\text{in } M))$ is a closed subset of Z which does not contain the point c. This contradicts the fact that $\overline{U}(\text{in } Z)=\varphi(\overline{V}(\text{in } M)) \ni c$. (The converse is not true in general; see Example below.)

Next, suppose that X is an F-space and $f \in C(Y)$ and $P(f) \smile N(f) \not\ni c$. f can be regarded as a function on $\varphi^{-1}(Y)$. By Lemma 2, $((\overline{P(f)}(\text{in } (Y)), \smile \overline{N(f)}(\text{in } \varphi^{-1}(Y))) \frown C=\theta$. Since C is a finite subset and $\varphi^{-1}(Y)$ is an F-space we can construct $g \in B(\varphi^{-1}(Y))$ such that $g(x)=-1$ on $\overline{P(f)}(\text{in } \varphi^{-1}(Y))$, $g(x)=1$ on $\overline{N(f)}(\text{in } \varphi^{-1}(Y))$, $g(x)=0$ on C and $-1 \leq g \leq 1$. Then we have $h=g\varphi^{-1} \in C(Y)$ because φ is a closed mapping and $\varphi(C)=c$. This shows that $P(f)$ and $N(f)$ are completely separated. In case X has the E_σ-property, our assertion will be obtained by an analogous method.

Theorem 7. *Suppose that* $C=\{a, b\}$. *Then we have* 1) *if* X *is* σ*-complete,* a *is a* P*-point of* M *and* b *is not a* P*-point of* M*, then* Z *is not* σ*-complete;* 2) *if* X *has the* E_σ*-property,* a *is a* P*-point of* M *and* b *is not a* P*-point of* M*, then any subspace* N *of* Z *containing* X *has the* E_σ*-property;* 3) *if* X *is an* F*-space and both* a *and* b *are not* P*-point of* M*, then* Z *is not an* F*-space.*

Proof. 1) There is $f \in B(M)$ such that $P(f) \not\ni b$ but $\overline{P(f)}(\text{in } M) \ni b$ and $P(f) \not\ni a$, $f(a)=0$. Since φ is a closed mapping, f can be regarded as a function on Z. Then $\overline{P(f)}(\text{in } Z) \ni c$ but c is not an inner point of $\overline{P(f)}(\text{in } Z)$ and hence Z is not σ-complete.

2) Let U be an F_σ-open subset of N, $f \in B(U)$. We regard f as a function in $B(\varphi^{-1}(U))$, and hence f has a continuous extension g over $\varphi^{-1}(N)$. If either i) $U \ni c$ or ii) $U \not\ni c$ and $g(a)=g(b)$, it is easy to see that g is considered as a continuous function on N. If $U \not\ni c$ and $g(a) \neq g(b)$, there is an open neighborhood W of a such that $W \frown \overline{\varphi^{-1}(U)}(\text{in } \varphi^{-1}(N))=\theta$ because a is a P-point of M. There exists $h \in B(\varphi^{-1}(N))$ such that $h(a)=1$, $h(x)=0$ on $\overline{\varphi^{-1}(U)}(\text{in } \varphi^{-1}(N))$ and $h(b)=0$. Then $k(x)=f(x)+f(b)h(x) \in B(\varphi^{-1}(N))$ and $k(a)=k(b)$, and hence k can be considered as a function in $B(Z)$ and $k|U=f$. This means that Z has the E_σ-property.

3) This follows from the fact that there exists $f \in B(M)$ such that $0 \leq f \leq 1$, $f(a)=f(b)=0$ and $\overline{P(f)}(\text{in } M) \ni a$, $\overline{N(f)}(\text{in } M) \ni b$.

Remark. It is easily seen that Theorems 6 and 7 are true if M is any space which contains X as a dense subset (but is not necessarily contained in βX) and any subspace containing X of which is an F-space or σ-complete or has the F_σ- or E_σ-property respectively.

Example. Let $L=[1, \Omega]$ be a space such that every point α ($\neq \Omega$) is an isolated point and a neighborhood of Ω is an interval in the usual sense. Then L is a normal P-space, and hence L has the F_σ-property. It is well known that Ω is a P-point of L [1, Example 8.7]. Therefore Ω is a P-point of βL by Lemma 1. Let $X=[1, \Omega)$ be a discrete space. Then βX has not P-points except points of X (see [4, 5.1] or [5, Th. 45]). The identical map of X onto X has a continuous extension φ of βX onto βL. We shall show that *if* $z \neq \Omega$, *then* $\varphi^{-1}(z)$ *consists of only one point, and* $\beta X - \varphi^{-1}(\Omega)$ *is homeomorphic to* $\beta L - \Omega$. Suppose that $z \neq \Omega$. Ω has a neighborhood U in βL which is disjoint from z and $L \frown U \supset \{\alpha; \alpha \geq \alpha_0\}$ for suitable ordinal α_0. $V = L - U \frown L$ is open and closed in X and in L. Hence βV is considered as an open and closed subset in βX and in βL. This shows that our assertions are true.

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