6. On Some Properties of Group Characters

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Let $\mathfrak{G}$ be a group of finite order and let p be a fixed prime number. An element is called a p-element of $\mathfrak{G}$ if its order is a power of p. An arbitrary element G of $\mathfrak{G}$ can be written uniquely as a product PR of two commutative elements where P is a p-element, while R is a p-regular element, i.e. an element whose order is prime to p. We shall call P the p-factor of G and R the p-regular factor of G. We define the section $\mathfrak{S}(P)$ of a p-element P as the set of all elements of $\mathfrak{G}$ whose p-factor is conjugate to P in $\mathfrak{G}$. Let $\mathfrak{K}_\nu$ be a class of conjugate elements which contains an element whose p-factor is P. Then $\mathfrak{S}(P)$ is the union of these classes $\mathfrak{K}_\nu$. Let $P_1=1, P_2,\cdots,P_h$ be a system of p-elements such that they all lie in different classes of conjugate elements, but that every p-element is conjugate to one of them. Then all elements of $\mathfrak{G}$ are distributed into h sections $\mathfrak{S}(P_i)$.

We consider the representations of $\mathfrak{G}$ in the field of all complex numbers. Let $\chi_1, \chi_2,\cdots,\chi_n$ be the distinct irreducible characters of $\mathfrak{G}$. Then the χ_i are distributed into a certain number of blocks $B_1, B_2,\cdots,B_t$. We denote by $\bar{a}$ the conjugate of a complex number a. Then $\bar{\chi}_i(G)=\chi_i(G^{-1})$. In [1] the following theorem has been stated without proof:

Let B be a block of $\mathfrak{G}$. If the elements G and H of $\mathfrak{G}$ belong to different sections of $\mathfrak{G}$, then

$$(1) \qquad \sum \chi_i(G)\bar{\chi}_i(H)=0$$

where the sum extends over all $\chi_i \in B$.

Recently the proof of this theorem was given in [2]. In this note, corresponding to the above theorem, we shall prove the following

Theorem 1. *Let $\mathfrak{S}(P)$ be a section of $\mathfrak{G}$. If the characters χ_i and χ_j belong to different blocks, then*

$$\sum{}' \chi_i(G)\bar{\chi}_j(G)=0$$

where the sum extends over all $G \in \mathfrak{S}(P)$.

As a consequence of Theorem 1, some new results are also obtained.

1. Let $\mathfrak{K}_\nu$ $(\nu=1,2,\cdots,n)$ be the classes of conjugate elements in $\mathfrak{G}$ and let G_ν be a representative of $\mathfrak{K}_\nu$. We shall first prove the following

Lemma. *If $\sum_{\nu=1}^{n} a_\nu\chi_i(G_\nu)=0$ for all $\chi_i \in B$, then $\sum_\alpha{}' a_\alpha\chi_i(G_\alpha)=0$ where the sum extends over all $\mathfrak{K}_\alpha \in \mathfrak{S}(P)$.*

Proof. Let $\mathfrak{K}_\beta$ be a class belonging to $\mathfrak{S}(P)$. We multiply by

$\overline{\chi}_i(G_\beta)$ and add over all $\chi_i \in B$. Using (1), we find

$$\sum_\alpha{}' a_\alpha \sum_{\chi_i \in B} \chi_i(G_\alpha)\overline{\chi}_i(G_\beta)=0.$$

Here we multiply by $\overline{a}_\beta$ and add over all $\mathfrak{K}_\beta \in \mathfrak{S}(P)$. Then

$$\sum_{\chi_i \in B} | \sum_\alpha{}' a_\alpha \chi_i(G_\alpha) |^2=0.$$

Hence we have for all $\chi_i \in B$

$$\sum_\alpha{}' a_\alpha \chi_i(G_\alpha)=0.$$

Denote by g_ν the number of elements in $\mathfrak{K}_\nu$. As is well known, we have the following character relations:

$$(2) \qquad \sum_\nu g_\nu \chi_i(G_\nu)\overline{\chi}_j(G_\nu)=0 \qquad (i \neq j),$$

and hence (2) is also valid for all $\chi_i \in B$ if $\chi_j \notin B$. As an application of Lemma, we obtain from (2) immediately

$$\sum_\alpha{}' g_\alpha \chi_i(G_\alpha)\overline{\chi}_j(G_\alpha)=0 \qquad (\chi_i \text{ and } \chi_j \text{ in different blocks}).$$

Hence Theorem 1 is proved.

Since the section $\mathfrak{S}(1)$ consists of all p-regular elements of $\mathfrak{G}$, it follows from Theorem 1 that

$$(3) \qquad \sum_R \chi_i(R)\overline{\chi}_j(R)=0 \qquad (\chi_i \text{ and } \chi_j \text{ in different blocks})$$

where R ranges over all p-regular elements of $\mathfrak{G}$. The relations (3) have been obtained in [4] by a different method. We may assume that the 1-character χ_1 belongs to B_1. If we set $\chi_j=\chi_1$ in Theorem 1, then we have

$$(4) \qquad \sum{}' \chi_i(G)=0 \qquad (\text{for } \chi_i \notin B_1)$$

where the sum extends over all $G \in \mathfrak{S}(P)$. In particular,

$$(5) \qquad \sum_R \chi_i(R)=0 \qquad (\text{for } \chi_i \notin B_1).$$

Theorem 2. *A character χ_i belongs to the first block B_1 if and only if $\sum_R \chi_i(R) \neq 0$.*

Proof. For every $\chi_i \in B_1$ we have, as was shown in [3]

$$\sum_R \chi_i(R)\overline{\chi}_1(R)=\sum_R \chi_i(R) \neq 0.$$

This, combined with (5), proves Theorem 2.

Let $\varphi_1, \varphi_2, \cdots, \varphi_m$ be the distinct modular irreducible characters of $\mathfrak{G}$ (for p). Then the φ_κ are also distributed into t blocks B_τ. If φ_κ belongs to a block B, then $\varphi_\kappa(R)$ can be written as a linear combination of $\chi_j(R) \in B$. It follows from (3) that

$$(6) \qquad \sum_R \overline{\varphi}_\kappa(R)\chi_i(R)=0 \qquad (\varphi_\kappa \text{ and } \chi_i \text{ in different blocks}).$$

In particular, for $i=1$, we have

$$(7) \qquad \sum_R \varphi_\kappa(R)=0 \qquad (\text{for } \varphi_\kappa \notin B_1).$$

Denote by $\Gamma(\mathfrak{G})$ the group ring of $\mathfrak{G}$ over the field of all complex numbers and by $\mathfrak{Z}$ the center of the group ring. Let K_ν be the sum of all elements in $\mathfrak{K}_\nu$. Every character χ_i determines a character ω_i of $\mathfrak{Z}$ which is given by $\omega_i(K_\nu)=g_\nu\chi_i(G_\nu)/z_i$ where $z_i=\chi_i(1)$. We may assume that $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_s$ are the classes belonging to $\mathfrak{S}(1)$, i.e. the p-regular classes. It follows from (6) that

$$\sum_{\alpha=1}^{s}\bar{\varphi}_\kappa(R_\alpha)\omega_i(K_\alpha)=0 \qquad (\varphi_\kappa \text{ and } \chi_i \text{ in different blocks}). \tag{8}$$

2. If P is an element of $\mathfrak{G}$ whose order is $p^a\geqq 1$ and if R is a p-regular element of the normalizer $\mathfrak{N}(P)$ of P, then we have

$$\chi_i(PR)=\sum_\kappa d_{i\kappa}^P\varphi_\kappa^P(R) \tag{9}$$

where the φ_κ^P are the modular irreducible characters of $\mathfrak{N}(P)$ and where the $d_{i\kappa}^P$ are algebraic integers of the field of the p^ath roots of unity. As was shown in [2], if we consider only χ_i belonging to a fixed block B_τ of $\mathfrak{G}$, then only characters φ_κ^P have to be taken which belong to a well-determined set of blocks B_σ^P of $\mathfrak{N}(P)$. We shall say that B_τ is the block of $\mathfrak{G}$ determined by blocks B_σ^P of $\mathfrak{N}(P)$. Every block B_σ^P of $\mathfrak{N}(P)$ determines uniquely a block of $\mathfrak{G}$.

Originally, only the ordinary characters χ_i of $\mathfrak{G}$ and the modular characters φ_κ of $\mathfrak{G}$ were distributed into blocks B_τ. It is now natural to count φ_κ^P as a character of B_τ, if φ_κ^P belongs to a block B_σ^P of $\mathfrak{N}(P)$ which determines B_τ. Denote by x_τ the number of $\chi_i\in B_\tau$ and by y_τ the number of $\varphi_\kappa\in B_\tau$. Then B_τ consists of x_τ ordinary characters and x_τ modular characters $\varphi_\kappa^{P_i}$. B_τ contains y_τ modular characters φ_κ of $\mathfrak{G}$ and the other $\varphi_\kappa^{P_i}$ are the modular characters of the normalizers $\mathfrak{N}(P_i)$.

Let $R_1, R_2, \cdots, R_l$ be a complete system of representatives for the p-regular classes of $\mathfrak{N}(P)$. Then the section $\mathfrak{S}(P)$ consists of l classes of conjugate elements and a complete system of representatives for these classes is given by PR_α $(\alpha=1, 2, \cdots, l)$. In the following we denote by $\mathfrak{K}_\alpha^P$ the class of $\mathfrak{G}$ which contains PR_α and by K_α^P the sum of all elements in $\mathfrak{K}_\alpha^P$.

Theorem 3. *If χ_i and φ_κ^P belong to different blocks, then*

$$\sum_{\alpha=1}^{l} g_\alpha^P\bar{\varphi}_\kappa^P(R_\alpha)\chi_i(PR_\alpha)=0$$

where g_α^P denotes the number of elements in $\mathfrak{K}_\alpha^P$.

Proof. If φ_κ^P belongs to a block B, then we see from (9) that $\varphi_\kappa^P(R_\alpha)$ can be written as a linear combination of $\chi_i(PR_\alpha)$ where $\chi_i\in B$. Hence Theorem 3 follows from Theorem 1 immediately.

Evidently Theorem 3 is a generalization of (6). We have from Theorem 3

$$\sum_{\alpha=1}^{l}\bar{\varphi}_\kappa^P(R_\alpha)\omega_i(K_\alpha^P)=0 \qquad (\chi_i \text{ and } \varphi_\kappa^P \text{ in different blocks}). \tag{10}$$

Denote by $\mathfrak{Z}^*$ the center of the modular group ring $\Gamma^*(\mathfrak{G})$ of $\mathfrak{G}$. Then $\mathfrak{Z}^*$ splits into a direct sum of t indecomposable ideals $\mathfrak{Z}_\tau^*$. Let $\mathfrak{Z}_\tau^*$ be the ideal corresponding to a block B_τ. Let $\zeta_1, \zeta_2, \cdots, \zeta_m$ be the modular irreducible characters of $\mathfrak{G}$ in the original sense, that is, $\zeta_\kappa(R)$ be the residue class of $\varphi_\kappa(R)$ (mod $\mathfrak{p}$) where $\mathfrak{p}$ denotes a suitable prime ideal divisor of p. If we set

$$C_\kappa^P = \sum_{\alpha=1}^{l} \zeta_\kappa^P(R_\alpha^{-1}) K_\alpha^P, \tag{11}$$

then (10) implies that $C_\kappa^P \in \mathfrak{Z}_\tau^*$ if and only if φ_κ^P belongs to B_τ. Since $|\varphi_\kappa^P(R_\alpha)| \not\equiv 0$ (mod $\mathfrak{p}$) for every P, we see that the C_κ^P form a basis of $\mathfrak{Z}^*$ and moreover the C_κ^P with $\varphi_\kappa^P \in B_\tau$ form a basis of $\mathfrak{Z}_\tau^*$.

Added in Proof. Professor R. Brauer communicated to me that he had also obtained Theorem 1 by a different method.

References

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