148. *The Diffusion Satisfying Wentzell's Boundary Condition and the Markov Process on the Boundary. II*

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1. In Part I [3], we have introduced a special class of Markov processes on the boundary with generator $\overline{LH}_\alpha$, $\alpha\geq 0$. These are used to obtain the diffusion on $\overline{D}$ determined by

$$(1)\qquad \frac{\partial}{\partial t}u(x)=Au(x),\quad x\in D,$$

$$(2)\qquad Lu(x)=0,\quad x\in\partial D.$$

Now, the problem is to *inquire the probabilistic meaning of these processes concerning the diffusion on $\overline{D}$.* Among them, we first consider the process for $\overline{LH}_0$, which seems to be most important by the following reason.

Roughly speaking, the system of resolvents $\{K_\beta,\ \beta>0\}$ for the process with generator $\overline{LH}_0$ is obtained by solving

$$(3)\qquad Au(x)=0,\quad x\in 0$$

$$(4)\qquad (\beta-L)u(x)=f(x),\quad x\in\partial D,$$

for given $f\in C(\partial D)$, and then defining

$$(5)\qquad K_\beta: f\longrightarrow K_\beta f=[u]_{\partial D}\in C(\partial D),$$

where $[u]_{\partial D}$ is the restriction on ∂D of the solution u of (3)–(4). Hence, there is a kind of "*duality*" in appearance between this operation of obtaining K_β and the following operation of obtaining the resolvents $\{G_\alpha,\ \alpha>0\}$ of the diffusion on $\overline{D}$. Solve

$$(4')\qquad (\alpha-A)u(x)=v(x),\quad x\in D$$

$$(3')\qquad Lu(x)=0,\quad x\in\partial D$$

for given $v\in C(\overline{D})$, and then define

$$(5')\qquad G_\alpha: v\longrightarrow G_\alpha v=u\in C(\overline{D}),$$

where u is the solution for (3′)–(4′).

The process on the boundary with generator $\overline{LH}_\alpha$, $\alpha>0$ can be considered to correspond with the diffusion on $\overline{D}$ determined by

$$(1')\qquad \frac{\partial}{\partial t}u(x)=(A-\alpha)u(x),\quad x\in D$$

$$(2)\qquad Lu(x)=0,\quad x\in\partial D.$$

2. *A probabilistic interpretation.* Now, we call the process on ∂D with generator $\overline{LH}_0$ *the Markov process on the boundary concerning*

the diffusion, and consider the following probabilistic interpretation of the process. *Assume that there lives an animal on the boundary, which is able to see only the sights on the boundary and is observing the particle of the diffusion determined by* (1) *and* (2). *Then, he observes a motion of the particle on the boundary, which is the trace on ∂D of the trajectory of the diffusion. This is the Markov process on the boundary concerning the diffusion.*

3. *A justification of the interpretation.* To justify the interpretation rigorously, we consider a special case. A little apart from the set up in Part I, take $D=\{x=(x_1,\cdots,x_N)\in R^N \mid x_N>0\}$ where R^N is the N-dimensional Euclidean space and let $C(\overline{D})$ and $C(\partial D)$ be the space of continuous functions defined on the one-point compactifications of $\overline{D}$ and ∂D respectively. The diffusion equation and the boundary condition are given by

$$\frac{\partial}{\partial t}u(x)=\Delta u(x), \quad x\in D \tag{6}$$

$$Lu(x)=\frac{\partial}{\partial n}u(x)=0, \quad x\in\partial D. \tag{7}$$

Then, there is the Brownian motion process $\{X(t,w),\ 0\leq t<\infty,\ w\in W\}$ defined on the space W of continuous functions $w(t)$ on $[0,\infty)$ taking values in $\overline{D}$, where $X(t,w)$ is the coordinate function of w at time t.

$$T_t u(x)=E_x(u(X(t,w))),\ u\in C(\overline{D}),\ t\geq 0$$

form a semigroup on $C(\overline{D})$, strongly continuous in t, with generator $\mathfrak{G}$, which is a contraction of the closure of Δ, and such that any smooth $u\in\mathfrak{D}(\mathfrak{G})$ satisfies (7). $E_x(\cdot)$ is the expectation with respect to the measure $P_x(\cdot)$ on $(W,\mathfrak{B})$, where $P_x(\cdot)$ gives the probability law of the motion under the condition that the particle starts at $x\in\overline{D}$, and $\mathfrak{B}$ is the smallest Borel field containing $\{w \mid w(t_i)\in A_i,\ 1\leq i\leq n\}$ for any open set $A_i\subset\overline{D}$.[1] Operators H_0 and $\overline{LH_0}=\overline{\frac{\partial}{\partial n}H_0}$ are defined similarly as in Part I–2, and there is the Markov process on the boundary concerning the Brownian motion, which defines a semigroup $\{\widetilde{T}_t\}$ with generator $\overline{\frac{\partial}{\partial n}H_0}$.

Since the set of times at which the particle stays on the boundary is measure 0, i.e. $\{t\geq 0 \mid X(t,w)\in\partial D\}$ is measure 0 with $P_x(\cdot)$-probability 1, we need a time scale suitable to describe the trace of the motion on the boundary. Noting that $X(t,w)$ can be considered as a vector

1) The precise definition of $P_x(\cdot)$ is referred to K. Ito and H. P. McKean [1].

whose components $X_i(t, w)$, $1 \leq i \leq N$ are mutually independent one-dimensional Brownian motions, and using a result of K. Ito and H. P. McKean [1] on one-dimensional local time, we know that

$$\mathfrak{t}(t, w) = \lim_{\rho \to 0} \frac{1}{\rho} \int_0^t \chi_{\bar{D}_\rho}(X(s, w))ds, \quad w \in W^{2)}$$

converges with $P_x(\cdot)$-probability 1, where $D_\rho = \{x = (x_1, \cdots, x_N) \in D \mid x_n < \rho\}$ and $\chi_{\bar{D}_\rho}$ is the characteristic function of $\bar{D}_\rho$. $\mathfrak{t}(t, w)$, called *local time on the boundary*, increases monotonically to ∞, when t tends to ∞ monotonically, and increases only when $X(t, w)$ is at the boundary with $P_x(\cdot)$-probability measure 1. Hence, we take the inverse function $\mathfrak{t}^{-1}(t, w)$ as the time scale to describe the trace of the path $w(t)$ on the boundary. Then, the motion on the boundary is represented by

$$X(\mathfrak{t}^{-1}(t, w)) \in \partial D \tag{8}$$

with respect to this time scale. Now, it is sufficient to prove

$$E_x\{f(X(\mathfrak{t}^{-1}(t, w)))\} = \tilde{T}_t f(x), \quad f \in C(\partial D), \quad x \in \partial D, \tag{9}$$

to justify the interpretation in 2.

To get (9) we use the idea of an interpretation of Cauchy process by F. Spitzer [2]. Let $\sigma(w) = \inf\{t \geq 0 \mid X(t, w) \in \partial D\}$ and put

$$\begin{aligned} \tilde{\tilde{T}}_t f(x) = E_{(x_1, \cdots, x_{N-1}, t)}\{f(X(\sigma(w), w))\}, \quad & f \in C(\partial D), \\ & x = (x_1, \cdots, x_{N-1}, 0) \in \partial D. \end{aligned} \tag{10}$$

$\{\tilde{\tilde{T}}_t\}$ forms a strongly continuous semigroup on $C(\partial D)$. Moreover, we know that $\tilde{T}_t = \tilde{\tilde{T}}_t$ since the right hand side of (10) is $H_0 f(x_1, \cdots, x_{N-1}, t)$ as a function of $(x_1, \cdots, x_{N-1}, t) \in \bar{D}$, and hence

$$\lim_{t \to 0} \frac{1}{t}(\tilde{\tilde{T}}_t f(x) - f(x)) = \frac{\partial}{\partial n} H_0 f(x) = \overline{\frac{\partial}{\partial n} H_0 f(x)}$$

when the left hand side converges strongly.

On the other hand, it holds that

$$E_{(x_1, \cdots, x_{N-1}, t)}(\sigma(w) \in [s_1, s_2)) = E_{(x_1, \cdots, x_{N-1}, 0)}(\mathfrak{t}^{-1}(t, w) \in [s_1, s_2)) \tag{11}$$

for any $[s_1, s_2) \subset [0, \infty)$ and $(x_1, \cdots, x_{N-1}, t) \in \bar{D}$ by a result on local time in one dimension, which implies

$$E_{(x_1, \cdots, x_{N-1}, 0)}\{f(X(\mathfrak{t}^{-1}(t, w)))\} = E_{(x_1, \cdots, x_{N-1}, t)}\{f(X(\sigma(w), w))\} = \tilde{T}_t f(x)$$

for $f \in C(\partial D)$ and $x = (x_1, \cdots, x_{N-1}, 0) \in C(\partial D)$. This completes the proof of (9).

4. *Comments on the general case.* In the proof of the justification in 3 we have used special tools closely connected with this special case in full. But the proof, especially the use of local time suggests a conjecture that in general case, in which the equation and the boundary condition are given by (1) and (2), *there is also a "local time on the*

2) This is essentially the one-dimensional local time of the N-th component of $X(t, w)$.

boundary" $\mathfrak{t}(t,w)$ suitably defined for the given diffusion and that *the Markov process on the boundary concerning the diffusion is explained by this local time similarly as in the special case in* 3, i.e. the process is represented by (8). If the conjecture is true, then the interpretation in 3 is completely justified.

A more rough justification will be given in the following way. Take $D_\rho=\{x\in D\mid \rho(x,\partial D)<\rho\}$ for $\rho>0$, where $\rho(x,\partial D)$ is the distance between x and ∂D. $X(t,w)$ is the diffusion given by (1) and (2) having right continuous path functions with left limits. Then, the system $\{G_\beta^{D_\rho},\beta>0\}$ given by

$$G_\beta^{D_\rho}u(x)=E_x\Big(\int_0^\infty e^{-\beta\int_0^t \chi_{\bar D_\rho}(X(s))ds}u(x(t))\chi_{\bar D_\rho}(X(t))dt\Big),\quad u\in C(\bar D)$$

determines a Markov process on the subset $\bar D_\rho$ of $\bar D$. $G_\beta^{D_\rho}u(x)$, for smooth $u\in C(\bar D)$, satisfies

$$(\beta-A)G_\beta^{D_\rho}u(x)=u(x)\quad\text{if}\quad x\in D_\rho,\quad =0\quad\text{if}\quad x\in D-\bar D_\rho.$$

Now, prove that $K_\beta[u]_{\partial D}$ is obtained as a limit of a sequence of some such quantities as $G_\beta^{D_\rho}u$ by letting $\rho\to 0$. This means an approximation of the process on the boundary by a sequence of processes on subsets $\bar D_\rho$. We note that in the special case in 3 such an approximation is also possible. In fact, instead of using $G_\beta^{D_\rho}u(x)$ directly, define

$$(12)\qquad G_\beta^\rho u(x)=E_x\Big(\int_0^\infty e^{-\beta\cdot\frac{1}{\rho}\int_0^t\chi_{\bar D_\rho}(X(s))ds}u(X(t))\chi_{\bar D_\rho}(X(t))\frac{dt}{\rho}\Big),\quad u\in C(\bar D),$$

which means the slowing down of the speed of original process ρ-times on $\bar D_\rho$. Then, rewriting (11) in the form

$$G_\beta^\rho u(x)=-\frac{1}{\beta}E_x\Big(\int_0^\infty u(x(t))\,d(e^{-\frac{\beta}{\rho}\int_0^t\chi_{\bar D_\rho}(X(s))ds})\Big)$$

we have

$$\lim_{\rho\to 0}G_\beta^\rho u(x)=-\frac{1}{\beta}E_x\Big(\int_0^\infty u(X(t))\,d(e^{-\beta\mathfrak{t}(t,w)})\Big)$$

$$=E_x\Big(\int_0^\infty e^{-\beta\mathfrak{t}(t,w)}u(X(t))\,d\mathfrak{t}(t,w)\Big)=E_x\Big(\int_0^\infty e^{-\alpha t}u(X(\mathfrak{t}^{-1}(t,w),w))\,dt\Big)$$

$$=E_x\Big(\int_0^\infty e^{-\beta t}[u]_{\partial D}(X(\mathfrak{t}^{-1}(t,w)))\,dt\Big)=\int_0^\infty e^{-\beta t}\widetilde T_t[u]_{\partial D}(x)dt=K_\beta[u]_{\partial D}(x),$$

completing the proof. In general case, suitable change of the speed on the subsets should be applied. We note that there are some other cases in which only this method is available at this stage.

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