7. *On Transformation of the Seifert Invariants*

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The theory of continuous transformations of manifolds shows preference to the case that $\dim X = \dim Y$ or $\dim X > \dim Y$ where X is mapped into Y. The reason is that every continuous mapping of an m-sphere into an n-sphere with $m < n$ is homotopic to zero. We will cast a look on the case $\dim X < \dim Y$.

1. Suppose z, z' are two disjoint zero-divisors in the compact manifold X such that $\dim z + \dim z' \geq (\dim X) - 1$. Then the pair (z, z') determines [1] a rational interlacing cycle, $\sigma(z, z')$, as follows. Let a, b be the smallest positive integers satisfying $az \sim 0$ and $bz' \sim 0$, and let A, B be two finite integral chains in X such that $\partial A = az$ and $\partial B = bz'$. Then, if f denotes the usual intersection function,

$$\frac{1}{a} f(A, z') = \frac{1}{ab} f(A, \partial B) = \pm \frac{1}{ab} f(\partial A, B) = \pm \frac{1}{ab} f(az, B) = \pm \frac{1}{b} f(z, B).$$

One thus obtains an expression that does not depend on A. Now

$$\sigma(z, z') = \frac{1}{a} f(A, z')$$

is Seifert's interlacing cycle.

2. Let $2 \leq m < n$ be integers, let M be an m-dimensional and N an n-dimensional oriented differentiable compact manifold, moreover $f: M \to N$ a continuous mapping. Let P, Q, R, S be pairwise disjoint oriented differentiable compact manifolds in N such that

$$p \geq n-m, \quad q \geq n-m, \quad r \geq n-m, \quad s \geq n-m,$$
$$p+q+r+s = 4n-m-3, \qquad p+q \geq 2n-m,$$

where p, q, r, s are the dimensions of P, Q, R, S respectively. For instance setting

$$p = q = r = n-1 \quad \text{and} \quad s = n-m,$$

one confirms at once that the above dimensional suppositions are fulfilled.

The algebraic inverse of P, Q, R, S under f, defined for instance in [4], will be denoted by z_P, z_Q, z_R, z_S respectively. Geometrically one can suppose [5] that the inverses of P, Q, R, S are differentiable manifolds. Then z_P, z_Q, z_R, z_S is an integral cycle of dimension $p-(n-m)$, $q-(n-m)$, $r-(n-m)$, and $s-(n-m)$ respectively. Let the manifolds P, Q, R, S be defined in such a way that z_P, z_Q, z_R, z_S are zero-divisors. That is always possible as one easily confirms. Let z_T denote the above defined Seifert interlacing cycle, $\sigma(z_P, z_Q)$. By

$$\begin{aligned}\dim z_T &= (\dim z_P) + 1 + \dim z_Q - \dim M \\ &= (p-n+m)+1+(q-n+m)-m = p+q-2n+m+1\end{aligned}$$

and the supposition $p+q\geq 2n-m$, it follows that $\dim z_T \geq 1$.

Let a, b, c be the smallest positive integers such that cz_T is an integral cycle and that moreover

$$az_R \sim 0 \quad \text{and} \quad bz_S \sim 0.$$

Let A, B be chains in M satisfying $\partial A = az_R$ and $\partial B = bz_S$. Furthermore let $Z_1, Z_2, \cdots$ be a base of the integral $(r+1)$-cycles in M and $Z'_1, Z'_2, \cdots$ be a base of the integral $(s+1)$-cycles in M. Now f being as above the intersection function, we set

$$\zeta_i = f(A+Z_i, cz_T),$$
$$\zeta_{ij} = f(\zeta_i, B+Z'_j).$$

Then

$$\begin{aligned}\dim \zeta_{ij} &= \dim \zeta_i + (\dim z_S) + 1 - \dim M \\ &= (\dim z_R) + 1 + \dim z_T - \dim M_S + (\dim z) + 1 - \dim M \\ &= (\dim z_R) + 1 + (\dim z_P) + 1 + \dim z_Q - \dim M - \dim M \\ &\quad + (\dim z_S) + 1 - \dim M \\ &= \dim z_P + \dim z_Q + \dim z_R + \dim z_S + 3 - 3 \dim M \\ &= p+q+r+s-4n-4m+3-3m = (4n-m-3)-4n+m+3 = 0.\end{aligned}$$

Thus the ζ_{ij} are integers. The matrix consisting of these numbers is invariant under deformation of f. In order that f is an essential map, it suffices that at least on ζ_{ij} is not zero. To the matrix (ζ_{ij}) there corresponds a comatrix that one obtains by projecting our results in the cohomology rings of M and N, see for instance [2, 3].

3. Let r be a positive integer $\leq m-1$ such that every integral homology class of dimension $n-r-1$ and likewise every such class of dimension $n-m+r$ of N permits a realization 3 by an oriented differentiable compact manifold. Now let the $(n-r-1)$-manifolds $A_1, A_2, \cdots$ and the $(n-m+r)$-manifolds $B_1, B_2, \cdots$ be bases of the integral $(n-r-1)$-cycles and the $(n-m+r)$-cycles of N. Let z_i, z'_i be the algebraic inverse of A_i and B_i respectively. Suppose that A_i and B_i are ordered in such a way that z_i is zero-divisor for $i=1, 2, \cdots, \alpha$ and only for these i's, and that z'_i is zero-divisor for $i=1, 2, \cdots, \beta$ and only for these i's. For all pairs (i, j) satisfying $i \leq \alpha$ and $j \leq \beta$, now let σ_{ij} be Seifert's interlacing number of (z_i, z'_j).

Then one again obtains a characteristic matrix (σ_{ij}) of f that possesses similar properties for the matrix of section 2.

References

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