# 52. On Outer Automorphisms of Certain Finite Factors

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1. In a recent paper [4], T. Saitô proved a remarkable theorem: If $A$ and $B$ are finite continuous factors, and if $G$ and $H$ are countable groups of automorphisms of $A$ and $B$ respectively, then one has (in the notation of [3])

$$(1)\qquad (G\times H)\otimes(A\otimes B)=(G\otimes A)\otimes(H\otimes B),$$

where the action of $g\otimes h$, with $g\in G$ and $h\in H$, on $A\otimes B$ is defined by

$$(2)\qquad (a\otimes b)^{g\otimes h}=a^g\otimes b^h,$$

(in (2), $a^g$ and $b^h$ indicate the actions of $g$ and $h$ on $a\in A$ and $b\in B$ respectively). Besides that Saitô gave an interrelation between the crossed and direct products of von Neumann algebras, it is remarkable that Saitô's theorem implies a theorem [4; Thm. 2], which may shed a light on the classifications of finite continuous factors in future.

However, in an approach of the crossed product of von Neumann algebras presented by the authors [3], it is unsatisfactory that Saitô remains to prove a fact that $G\times H$ is a group of outer automorphisms of $A\otimes B$ if $G$ and $H$ are groups of outer automorphisms of $A$ and $B$ respectively. The purpose of the present short note is to supply it by a help of a classical technique due to Murray and von Neumann [2].

2. The precise statement is as follows:

THEOREM. *If $g$ and $h$ are automorphisms on finite continuous factors $A$ and $B$ respectively and at least one of them is outer, then $g\otimes h$ defined by (2) is an outer automorphism of $A\otimes B$.*

Proof. It is sufficient by [1; Chap. 1, § 4, Prop. 2] that $g\otimes h$ is outer on $A\otimes B$. To prove, it is not less general to assume that $g$ is outer. If $g\otimes h$ is inner on $A\otimes B$, then there is a unitary operator $u$ in $A\otimes B$ such that

$$(3)\qquad (a\otimes 1)u=u(a^g\otimes 1).$$

Now, by a classical argument due to Murray and von Neumann [2; Chap. II], each operator in (3) can be described by a matrix with entries belonging to $A$:

$$a\otimes 1\sim\begin{pmatrix} a & 0 & 0 & \cdots \\ 0 & a & 0 & \cdots \\ 0 & 0 & a & \cdots \\ & \cdots & & \cdots \end{pmatrix},\qquad a^g\otimes 1\sim\begin{pmatrix} a^g & 0 & 0 & \cdots \\ 0 & a^g & 0 & \cdots \\ 0 & 0 & a^g & \cdots \\ & \cdots & & \cdots \end{pmatrix},$$

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and

$$u \sim \begin{pmatrix} u_{11} & u_{12} & u_{13} \cdots \\ u_{21} & u_{22} & u_{23} \cdots \\ u_{31} & u_{32} & u_{33} \cdots \\ & \cdots & \cdots \end{pmatrix}.$$

Computing both sides of (3), one has easily

$$au_{ij} = u_{ij}a^g, \tag{4}$$

for any $a \in A$ and $i, j = 1, 2, \cdots$ . Since $g$ is outer and $A$ is a finite continuous factor, (4) implies at once by [3; Lemma 1] $u_{ij} = 0$ for all $i$ and $j$, which is a contradiction.

## References

[1] J. Dixmier: Les Algèbres d'Opérateurs dans l'Espace Hilbertien, Gauthier-Villars, Paris (1957).

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