69. On Köthe's Problem concerning Algebras for which Every Indecomposable Module Is Cyclic. II

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This is a continuation of the previous paper with the same title which will be referred to as Part I. Throughout this paper, A will be assumed to be a ring which has a unit and satisfies the minimum condition for left ideals, and we shall use the same notation as in Part I.

§ 4. **Classification of quasi-primitive modules.** Let A satisfy the conditions (a°) and (b°) stated in § 3 of Part I. Then all quasi-primitive left A-modules are classified into the following types:

Type I: $Ae_\lambda g$ itself is uni-serial.

Type II: $Ae_\lambda g$ is a module such that $Ae_\lambda g/N^l e_\lambda g (l\geqq 1)$ is uni-serial, $N^l e_\lambda g = Ae_\kappa te_\lambda g \oplus Ae_\rho we_\lambda g (t, w \text{ in } N^l)$ where $Ae_\kappa te_\lambda g$ as well as $Ae_\rho we_\lambda g$ is uni-serial, and $S(Ae_\lambda g) = N^m e_\kappa te_\lambda g \oplus N^n e_\rho we_\lambda g (m\geqq 0,\ n\geqq 0)$. In particular, if $l\geqq 3$ then $m=n=0$ holds, and if $l=2$ then either $m=0$ or $n=0$ holds.

Type III: $Ae_\lambda g$ is a module such that $Ae_\lambda g/N^l e_\lambda g (l\geqq 1)$ is uni-serial, $N^l e_\lambda g = Ae_\kappa te_\lambda g + Ae_\rho we_\lambda g (t, w \text{ in } N^l)$ where $Ae_\kappa te_\lambda g$ as well as $Ae_\rho we_\lambda g$ is uni-serial, $Ae_\kappa te_\lambda g \frown Ae_\rho we_\lambda g = N^m e_\kappa te_\lambda g = N^n e_\rho we_\lambda g = Ae_\sigma ue_\rho we_\lambda g (m\geqq 1,\ n\geqq 1, u \text{ in } N)$, and $S(Ae_\lambda g) = N^k e_\sigma uwe_\lambda g (k\geqq 0)$. In particular, if $l\geqq 3$ (resp. $k\geqq 2$) then $m=n=1$ holds, and if $l=2$ (resp. $k=1$) then either $m=1$ or $n=1$ holds.

Type IV: $Ae_\lambda g$ is a module such that $Ae_\lambda g/N^l e_\lambda g (l=1 \text{ or } 2)$ is uni-serial, $N^l e_\lambda g = Ae_\kappa te_\lambda g + Ae_\rho we_\lambda g (t, w \text{ in } N^l)$ where $Ae_\kappa te_\lambda g$ is uni-serial, $Ae_\kappa te_\lambda g \frown Ae_\rho we_\lambda g = N^m e_\kappa te_\lambda g = Ae_\sigma ue_\rho we_\lambda g (m\geqq 1, u \text{ in } N)$, $Ne_\rho we_\lambda g = Ae_\sigma ue_\rho we_\lambda g \oplus Ae_\tau ve_\rho we_\lambda g (v \text{ in } N)$ where $Ae_\tau vwe_\lambda g$ is uni-serial, and $S(Ae_\lambda g) = Ae_\sigma uwe_\lambda g \oplus N^k e_\tau vwe_\lambda g (k\geqq 0)$. In particular, if $l=2$ then both $m=1$ and $k=0$ hold.

Type V: $Ae_\lambda g$ is a module such that $Ae_\lambda g/N^l e_\lambda g (l=1 \text{ or } 2)$ is uni-serial, $N^l e_\lambda g = Ae_\kappa te_\lambda g + Ae_\rho we_\lambda g (t, w \text{ in } N^l)$ where $Ae_\kappa te_\lambda g$ is uni-serial, $Ae_\kappa te_\lambda g \frown Ae_\rho we_\lambda g = N^m e_\kappa te_\lambda g = Ae_\sigma ue_\rho we_\lambda g (m\geqq 1, u \text{ in } N)$, $Ne_\rho we_\lambda g = Ae_\sigma ue_\rho we_\lambda g + Ae_\tau ve_\rho we_\lambda g (v \text{ in } N)$ where $Ae_\tau vwe_\lambda g$ is uni-serial, $Ae_\sigma uwe_\lambda g \frown Ae_\tau vwe_\lambda g = N^n e_\tau vwe_\lambda g = Ne_\sigma uwe_\lambda g = Ae_\mu se_\sigma uwe_\lambda g (n\geqq 1, s \text{ in } N)$ and $S(Ae_\lambda g) = N^k e_\mu suwe_\lambda g (k=0 \text{ or } 1)$. In particular, if $l=2$ (resp. $k=1$) then $m=n=1$ holds.

§ 5. **Classification of indecomposable modules.** For the sake of brevity, in the later statement we shall adopt the following notation: Let $Ae_\lambda g$ be a quasi-primitive left A-module and $Ae_\mu ne_\lambda g$

(n in N) a quasi-primitive submodule. Then by the cover of $Ae_{\mu}ne_{\lambda}g$ in $Ae_{\lambda}g$ we shall mean a submodule of $Ae_{\lambda}g$, generated by all the elements $e_{\tau}xe_{\lambda}g(e_{\tau}x$ in $A)$ satisfying the property that $Ne_{\tau}xe_{\lambda}g\supset Ae_{\mu}ne_{\lambda}g$ but $N^2e_{\tau}xe_{\lambda}g\not\supset Ae_{\mu}ne_{\lambda}g$. We always denote by $C_{Ae_{\lambda}g}(Ae_{\mu}ne_{\lambda}g)$(abr. $C(Ae_{\mu}ne_{\lambda}g)$) the cover of $Ae_{\mu}ne_{\lambda}g$ in $Ae_{\lambda}g$.

In case A satisfies the conditions (a°) and (b°) stated in §3 of Part I, all finitely generated indecomposable left A-modules are classified into the following types:[1]

I–1: $\mathfrak{N}=Ae_{\lambda}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type I, i.e. a uniserial module.

I–2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are both non-simple module of Type I such that $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=S(Ae_{\lambda_1}g_1)=S(Ae_{\lambda_2}g_2)$ and the isomorphism: $S(Ae_{\lambda_1}g_1)\rightarrow S(Ae_{\lambda_2}g_2)$ is maximal (and so $Ae_{\lambda_1}\not\approx Ae_{\lambda_2}$, $C(S(Ae_{\lambda_1}g_1))/S(Ae_{\lambda_1}g_1)\not\approx C(S(Ae_{\lambda_2}g_2))/S(Ae_{\lambda_2}g_2)$ and $S(\mathfrak{N})=S(Ae_{\lambda_1}g_1)$).

I–3·1: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is a module of Type I such that $Ne_{\lambda_1}g_1=Ae_{\rho}we_{\lambda_1}g_1$ (w in N), $S(Ae_{\lambda_1}g_1)=N^2e_{\lambda_1}g_1=Ae_{\sigma}ve_{\rho}we_{\lambda_1}g_1(v$ in $N)$, and $Ae_{\lambda_2}g_2$ is a module of Type I such that there exists a maximal monomorphism ψ: $Ne_{\lambda_1}g_1\rightarrow Ne_{\lambda_2}g_2$, $\psi(e_{\rho}we_{\lambda_1}g_1)=e_{\rho}re_{\lambda_2}g_2$ (r in N) (and so $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$, $C(Ae_{\rho}re_{\lambda_2}g_2)/Ae_{\rho}re_{\lambda_2}g_2\approx Ae_{\lambda_1}/Ne_{\lambda_1}$, $S(Ae_{\lambda_2}g_2)=Ae_{\sigma}ve_{\rho}re_{\lambda_2}g_2$ and $Ae_{\sigma}\not\approx Ae_{\rho}$), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_{\rho}we_{\lambda_1}g_1$, $e_{\rho}we_{\lambda_1}g_1=e_{\rho}re_{\lambda_2}g_2$ (and hence $S(\mathfrak{N})=Ae_{\sigma}vwe_{\lambda_1}g_1$).

I–3·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are respectively the same as in I–3·1, but $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_{\sigma}vwe_{\lambda_1}g_1$, $e_{\sigma}vwe_{\lambda_1}g_1=e_{\sigma}vre_{\lambda_2}g_2$ (and so $S(\mathfrak{N})=Ae_{\sigma}vwe_{\lambda_1}g_1\oplus Ae_{\rho}(we_{\lambda_1}g_1-re_{\lambda_2}g_2)$).

I–3·3: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2+Ae_{\lambda_3}g_3$, where $Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$ is the same as in I–3·2; that is, $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_{\sigma}vwe_{\lambda_1}g_1$, $e_{\sigma}vwe_{\lambda_1}g_1=e_{\sigma}vre_{\lambda_2}g_2$, and $Ae_{\lambda_3}g_3$ is a non-simple module of Type I such that there exists a monomorphism: $Ae_{\lambda_3}g_3\rightarrow Ne_{\lambda_2}g_2/Ae_{\sigma}vre_{\lambda_2}g_2$ (and so r in N^2, $Ae_{\lambda_3}\not\approx Ae_{\lambda_1}$, $Ae_{\lambda_3}\not\approx Ae_{\lambda_2}$), and $Ae_{\lambda_3}g_3\frown(Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2)=S(Ae_{\lambda_3}g_3)=Ae_{\rho}te_{\lambda_3}g_3(t$ in $N)$, $e_{\rho}te_{\lambda_3}g_3=e_{\rho}we_{\lambda_1}g_1-e_{\rho}re_{\lambda_2}g_2$(and hence $S(\mathfrak{N})=Ae_{\sigma}vwe_{\lambda_1}g_1\oplus Ae_{\rho}te_{\lambda_3}g_3$).

I–4·1: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is a module of Type I such that $Ne_{\lambda_1}g_1=Ae_{\rho}we_{\lambda_1}g_1$ (w in N) and $S(Ae_{\lambda_1}g_1)=Ae_{\sigma}pe_{\rho}we_{\lambda_1}g_1$ (p in N^2), and $Ae_{\lambda_2}g_2$ is a module of Type I such that $Ne_{\lambda_2}g_2=Ae_{\rho}re_{\lambda_2}g_2$ (r in N), $Ne_{\lambda_2}g_2\approx Ne_{\lambda_1}g_1$ but $Ae_{\lambda_2}g_2\not\approx Ae_{\lambda_1}g_1$, i.e. $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$ (and so $S(Ae_{\lambda_2}g_2)=Ae_{\sigma}pe_{\rho}re_{\lambda_2}g_2$), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_{\rho}we_{\lambda_1}g_1$, $e_{\rho}we_{\lambda_1}g_1=e_{\rho}re_{\lambda_2}g_2$ (and hence $S(\mathfrak{N})=Ae_{\sigma}pwe_{\lambda_1}g_1$).

I–4·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are respectively the same as in I–4·1, but $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_{\tau}ve_{\rho}we_{\lambda_1}g_1\neq 0$ (v in N), $e_{\tau}vwe_{\lambda_1}g_1=e_{\tau}vre_{\lambda_2}g_2$ (and hence if we put $C(Ae_{\tau}vwe_{\lambda_1}g_1)=Ae_{\mu}qe_{\rho}we_{\lambda_1}g_1$ and $C(Ae_{\tau}vre_{\lambda_2}g_2)=Ae_{\mu}qe_{\rho}re_{\lambda_2}g_2$ (r in A), then $S(\mathfrak{N})=Ae_{\sigma}pwe_{\lambda_1}g_1\oplus Ae_{\mu}(qwe_{\lambda_1}g_1-qre_{\lambda_2}g_2)$ and $Ae_{\sigma}\not\approx Ae_{\mu}$).

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1) In this section we shall always denote by $\mathfrak{N}$ a finitely generated indecomposable left A-module.

II–1·1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type II such that $Ae_{\lambda_1}g_1/N^le_{\lambda_1}g_1(l\geqq 3)$ is uni-serial, $N^{l-1}e_{\lambda_1}g_1=Ae_\mu se_{\lambda_1}g_1$ (s in N^{l-1}) and $S(Ae_{\lambda_1}g_1)=N^le_{\lambda_1}g_1=Ae_\kappa te_\mu se_{\lambda_1}g_1\oplus Ae_\rho we_\mu se_{\lambda_1}g_1$ (t, w in N) (and of course $Ae_\kappa\not\approx Ae_\rho$).

II–1·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in II–1·1, and $Ae_{\lambda_2}g_2$ is a non-simple module of Type I such that there exists a monomorphism: $Ae_{\lambda_2}g_2\rightarrow Ne_{\lambda_1}g_1/Ae_\kappa tse_{\lambda_1}g_1$ (and so $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\rho wse_{\lambda_1}g_1$, $e_\rho wse_{\lambda_1}g_1=e_\rho we_\mu pe_{\lambda_2}g_2$ (p in N) (and hence $S(\mathfrak{N})=Ae_\kappa tse_{\lambda_1}g_1\oplus Ae_\rho wse_{\lambda_1}g_1$).

II–1·3: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in II–1·1, and $Ae_{\lambda_2}g_2$ is a module of Type I such that $Ne_{\lambda_2}g_2\approx Ne_{\lambda_1}g_1/Ae_\kappa tse_{\lambda_1}g_1$ but $Ae_{\lambda_2}g_2\not\approx Ae_{\lambda_1}g_1/Ae_\kappa tse_{\lambda_1}g_1$, i.e. $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$, and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\rho wse_{\lambda_1}g_1$, $e_\rho wse_{\lambda_1}g_1=e_\rho we_\mu qe_{\lambda_2}g_2$($q$ in N^{l-1})(and so $S(\mathfrak{N})=Ae_\kappa tse_{\lambda_1}g_1\oplus Ae_\rho wse_{\lambda_1}g_1$).

II–2·1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type II such that $Ae_{\lambda_1}g_1/N^2e_{\lambda_1}g_1$ is uni-serial, $Ne_{\lambda_1}g_1=Ae_\mu se_{\lambda_1}g_1$ (s in N), $N^2e_{\lambda_1}g_1=Ae_\kappa te_\mu se_{\lambda_1}g_1\oplus Ae_\rho we_\mu se_{\lambda_1}g_1$ (t, w in N), $S(Ae_{\lambda_1}g_1)=Ae_\kappa tse_{\lambda_1}g_1\oplus S(Ae_\rho wse_{\lambda_1}g_1)$ and $S(Ae_\rho wse_{\lambda_1}g_1)=Ae_\nu qe_\rho wse_{\lambda_1}g_1$($q$ in A)(and of course $Ae_\kappa\not\approx Ae_\nu$).

II–2·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in II–2·1, and $Ae_{\lambda_2}g_2$ is a module of Type I such that $Ae_{\lambda_2}g_2\approx Ae_\mu se_{\lambda_1}g_1/Ae_\kappa tse_{\lambda_1}g_1$ (and so $Ae_{\lambda_2}\approx Ae_\mu$ and $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\rho wse_{\lambda_1}g_1$, $e_\rho wse_{\lambda_1}g_1=e_\rho we_\mu\zeta e_{\lambda_2}g_2$ (ζ in A, but not in N)(and hence $S(\mathfrak{N})=Ae_\kappa tse_{\lambda_1}g_1\oplus Ae_\nu qwse_{\lambda_1}g_1$).

II–2·3: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are respectively the same as in II–2·2, but $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\sigma pe_\rho wse_{\lambda_1}g_1\neq 0$ (p in N) (and so q in N), $e_\sigma pwse_{\lambda_1}g_1=e_\sigma pwe_\mu\zeta e_{\lambda_2}g_2$ (ζ in A, but not in N) (and hence if we put $C(Ae_\sigma pwse_{\lambda_1}g_1)=Ae_\tau ve_\rho wse_{\lambda_1}g_1$ and $C(Ae_\sigma pw\zeta e_{\lambda_2}g_2)=Ae_\tau ve_\rho w\zeta e_{\lambda_2}g_2$($v$ in A), then $S(\mathfrak{N})=Ae_\nu qwse_{\lambda_1}g_1\oplus Ae_\tau(vwse_{\lambda_1}g_1-vw\zeta e_{\lambda_2}g_2)$ and $Ae_\nu\not\approx Ae_\tau$).

II–2·4: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in II–2·1, and $Ae_{\lambda_2}g_2$ is a module of Type I such that there exists a maximal monomorphism ψ: $Ne_{\lambda_1}g_1/Ae_\rho wse_{\lambda_1}g_1\rightarrow Ne_{\lambda_2}g_2$, $\psi(e_\mu se_{\lambda_1}g_1)=e_\mu re_{\lambda_2}g_2$($r$ in N), $S(Ae_{\lambda_2}g_2)=Ae_\kappa te_\mu re_{\lambda_2}g_2$ (and so $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$, $C(Ae_\mu re_{\lambda_2}g_2)/Ae_\mu re_{\lambda_2}g_2\not\approx Ae_{\lambda_1}/Ne_{\lambda_1}$), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\kappa tse_{\lambda_1}g_1$, $e_\kappa tse_{\lambda_1}g_1=e_\kappa tre_{\lambda_2}g_2$ (and hence $S(\mathfrak{N})=Ae_\kappa tse_{\lambda_1}g_1\oplus Ae_\rho wse_{\lambda_1}g_1$).

II–3·1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type II such that $Ne_{\lambda_1}g_1=Ae_{\kappa_1}t_1e_{\lambda_1}g_1\oplus Ae_{\rho_1}w_1e_{\lambda_1}g_1$ (t_1, w_1 in N), $S(Ae_{\kappa_1}t_1e_{\lambda_1}g_1)=Ae_{\sigma_1}u_1e_{\kappa_1}t_1e_{\lambda_1}g_1$ (u_1 in A), $S(Ae_{\rho_1}w_1e_{\lambda_1}g_1)=Ae_{\sigma_2}v_1e_{\rho_1}w_1e_{\lambda_1}g_1$($v_1$ in A) (and so $S(\mathfrak{N})=Ae_{\sigma_1}u_1t_1e_{\lambda_1}g_1\oplus Ae_{\sigma_2}v_1w_1e_{\lambda_1}g_1$, $Ae_{\sigma_1}\not\approx Ae_{\sigma_2}$).

II–3·2: $\mathfrak{N}=\sum_{i=1}^{s}Ae_{\lambda_i}g_i(s\geqq 2)$, where for each $i(1\leqq i\leqq s)$ $Ae_{\lambda_i}g_i$ is a module of Type II such that $Ne_{\lambda_i}g_i=Ae_{\kappa_i}t_ie_{\lambda_i}g_i\oplus Ae_{\rho_i}w_ie_{\lambda_i}g_i$ (t_i, w_i in N), $S(Ae_{\kappa_i}t_ie_{\lambda_i}g_i)=Ae_{\sigma_i}u_ie_{\kappa_i}t_ie_{\lambda_i}g_i$($u_i$ in A), $S(Ae_{\rho_i}w_ie_{\lambda_i}g_i)=Ae_{\sigma_{i+1}}v_ie_{\rho_i}w_ie_{\lambda_i}g_i$ (v_i in A), and they possess the property such that $Ae_{\lambda_i}\not\approx Ae_{\lambda_j}$ if $i\neq j$, Ae_{σ_i}

$\not\approx Ae_{\sigma_j}$ if $i\neq j$, $Ae_{\lambda_i}g_i \frown Ae_{\lambda_{i+1}}g_{i+1}=Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i$, $e_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i=e_{\sigma_{i+1}}u_{i+1}t_{i+1}e_{\lambda_{i+1}}g_{i+1}$ for every $i\ (\leqq s-1)$, and that each homomorphism φ_i: $Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i \rightarrow Ae_{\sigma_{i+1}}u_{i+1}t_{i+1}e_{\lambda_{i+1}}g_{i+1}$ as well as $\varphi_i^{-1}(1\leqq i\leqq s-1)$ is maximal, i.e. $C(Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i)/N(C(Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i))\not\approx C(Ae_{\sigma_{i+1}}u_{i+1}t_{i+1}e_{\lambda_{i+1}}g_{i+1})/N(C(Ae_{\sigma_{i+1}}u_{i+1}t_{i+1}e_{\lambda_{i+1}}g_{i+1}))$ for every $i\ (\leqq s-1)$(and so $S(\mathfrak{N})=Ae_{\sigma_1}u_1t_1e_{\lambda_1}g_1\oplus\sum_{i=1}^{s}{}^{\circ}Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i$).[2]

II–3·3: $\mathfrak{N}=\sum_{i=1}^{s}Ae_{\lambda_i}g_i(s\geqq 2)$, where for each $i\ (1\leqq i\leqq s-1)\ Ae_{\lambda_i}g_i$ is a module of Type II such that $Ne_{\lambda_i}g_i=Ae_{\kappa_i}t_ie_{\lambda_i}g_i\oplus Ae_{\rho_i}w_ie_{\lambda_i}g_i\ (t_i,\ w_i$ in $N)$, $S(Ae_{\kappa_i}t_ie_{\lambda_i}g_i)=Ae_{\sigma_i}u_ie_{\kappa_i}t_ie_{\lambda_i}g_i(u_i$ in $A)$, $S(Ae_{\rho_i}w_ie_{\lambda_i}g_i)=Ae_{\sigma_{i+1}}v_ie_{\rho_i}w_ie_{\lambda_i}g_i$ $(v_i$ in $A)$, and $Ae_{\lambda_s}g_s$ is a module of Type I such that $Ne_{\lambda_s}g_s=Ae_{\kappa_s}t_se_{\lambda_s}g_s$ $(t_s$ in $N)$ and $S(Ae_{\lambda_s}g_s)=Ae_{\sigma_s}u_se_{\kappa_s}t_se_{\lambda_s}g_s(u_s$ in $A)$, and they possess the same property as in II–3·2 (and so $S(\mathfrak{N})=\sum_{i=1}^{s}{}^{\circ}Ae_{\sigma_i}u_it_ie_{\lambda_i}g_i$).

II–3·4: $\mathfrak{N}=\sum_{i=1}^{s}Ae_{\lambda_i}g_i\ (s\geqq 3)$, where for each $i\ (2\leqq i\leqq s-1)\ Ae_{\lambda_i}g_i$ is a module of Type II such that $Ne_{\lambda_i}g_i=Ae_{\kappa_i}t_ie_{\lambda_i}g_i\oplus Ae_{\rho_i}w_ie_{\lambda_i}g_i(t_i,\ w_i$ in $N)$, $S(Ae_{\kappa_i}t_ie_{\lambda_i}g_i)=Ae_{\sigma_i}u_ie_{\kappa_i}t_ie_{\lambda_i}g_i\ (u_i$ in $A)$, $S(Ae_{\rho_i}w_ie_{\lambda_i}g_i)=Ae_{\sigma_{i+1}}v_ie_{\rho_i}w_ie_{\lambda_i}g_i$ $(v_i$ in $A)$, and both $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_s}g_s$ are modules of Type I such that $Ne_{\lambda_1}g_1=Ae_{\rho_1}w_1e_{\lambda_1}g_1\ (w_1$ in $N)$, $S(Ae_{\lambda_1}g_1)=Ae_{\sigma_2}v_1e_{\rho_1}w_1e_{\lambda_1}g_1\ (v_1$ in $A)$ and $Ne_{\lambda_s}g_s=Ae_{\kappa_s}t_se_{\lambda_s}g_s\ (t_s$ in $N)$, $S(Ae_{\lambda_s}g_s)=Ae_{\sigma_s}u_se_{\kappa_s}t_se_{\lambda_s}g_s\ (u_s$ in $A)$ respectively, and they possess the same property as in II–3·2 (and so $S(\mathfrak{N})=\sum_{i=1}^{s-1}{}^{\circ}Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i$).

III–1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type III such that $Ae_{\lambda_1}g_1/N^le_{\lambda_1}g_1(l\geqq 2)$ is uni-serial, $N^le_{\lambda_1}g_1=Ae_{\kappa}te_{\lambda_1}g_1+Ae_{\rho}we_{\lambda_1}g_1(t,\ w$ in $N)$, and $Ae_{\kappa}te_{\lambda_1}g_1\frown Ae_{\rho}we_{\lambda_1}g_1=Ne_{\kappa}te_{\lambda_1}g_1\ (\subset Ne_{\rho}we_{\lambda_1}g_1)$ (and so $S(\mathfrak{N})=S(Ne_{\kappa}te_{\lambda_1}g_1)$).

III–2: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type III such that $Ne_{\lambda_1}g_1=Ae_{\kappa}te_{\lambda_1}g_1+Ae_{\rho}we_{\lambda_1}g_1\ (t,\ w$ in $N)$, $Ae_{\kappa}te_{\lambda_1}g_1\frown Ae_{\rho}we_{\lambda_1}g_1=Ne_{\rho}we_{\lambda_1}g_1=Ne_{\kappa}te_{\lambda_1}g_1=Ae_{\sigma}ue_{\kappa}te_{\lambda_1}g_1(u$ in $N)$ and $S(Ae_{\lambda_1}g_1)=N^ke_{\sigma}ute_{\lambda_1}g_1\ (k\geqq 2)$.

III–3·1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type III such that $Ne_{\lambda_1}g_1=Ae_{\kappa}te_{\lambda_1}g_1+Ae_{\rho}we_{\lambda_1}g_1\ (t,\ w$ in $N)$, $Ae_{\kappa}te_{\lambda_1}g_1\frown Ae_{\rho}we_{\lambda_1}g_1=Ne_{\kappa}te_{\lambda_1}g_1=Ae_{\sigma}ue_{\kappa}te_{\lambda_1}g_1\ (u$ in $N)$, $e_{\sigma}ue_{\kappa}te_{\lambda_1}g_1=e_{\sigma}se_{\rho}we_{\lambda_1}g_1\ (s$ in $N)$ and $S(Ae_{\lambda_1}g_1)=Ne_{\sigma}ute_{\lambda_1}g_1=Ae_{\tau}ve_{\sigma}ute_{\lambda_1}g_1(v$ in $N)$.

III–3·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is a module of Type III such that $Ne_{\lambda_1}g_1=Ae_{\kappa}te_{\lambda_1}g_1+Ae_{\rho}we_{\lambda_1}g_1(t,\ w$ in $N)$, $Ae_{\kappa}te_{\lambda_1}g_1\frown Ae_{\rho}we_{\lambda_1}g_1=Ne_{\rho}we_{\lambda_1}g_1=Ne_{\kappa}te_{\lambda_1}g_1=Ae_{\sigma}ue_{\kappa}te_{\lambda_1}g_1(u$ in $N)$, $e_{\sigma}ue_{\kappa}te_{\lambda_1}g_1=e_{\sigma}se_{\rho}we_{\lambda_1}g_1(s$ in $N)$ and $S(Ae_{\lambda_1}g_1)=Ne_{\sigma}ute_{\lambda_1}g_1=Ae_{\tau}ve_{\sigma}ute_{\lambda_1}g_1(v$ in $N)$, and $Ae_{\lambda_2}g_2$ is a module of Type I such that $Ne_{\lambda_2}g_2=Ae_{\kappa}re_{\lambda_2}g_2\approx Ae_{\kappa}te_{\lambda_1}g_1(r$ in $N)$ but $Ae_{\lambda_2}g_2\not\approx Ae_{\lambda_1}g_1$, i.e. $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$ (and so $Ae_{\tau}\not\approx Ae_{\sigma}$, $Ae_{\tau}\not\approx Ae_{\kappa}$), $N^2e_{\lambda_2}g_2=Ae_{\sigma}ure_{\lambda_2}g_2$ and $S(Ae_{\lambda_2}$

2) By $\sum_{i=1}^{s}{}^{\circ}\ \mathfrak{m}_i$ we shall imply a direct sum of $\mathfrak{m}_i\ (i=1,2,\cdots,s)$.

$g_2)=N^3e_{\lambda_2}g_2=Ae_\tau vure_{\lambda_2}g_2$, and further $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\kappa te_{\lambda_1}g_1$, $e_\kappa te_{\lambda_1}g_1=e_\kappa re_{\lambda_2}g_2$(and so $S(\mathfrak{N})=Ae_\tau vute_{\lambda_1}g_1$).

III–3·3: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are respectively the same as in III–3·2, but $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\sigma ute_{\lambda_1}g_1$, $e_\sigma ute_{\lambda_1}g_1=e_\sigma ure_{\lambda_2}g_2$ (and so $S(\mathfrak{N})=Ae_\tau vute_{\lambda_1}g_1\oplus Ae_\kappa(te_{\lambda_1}g_1-re_{\lambda_2}g_2)$).

III–3·4: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are respectively the same as in III–3·2, but $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\tau vute_{\lambda_1}g_1$, $e_\tau vute_{\lambda_1}g_1=e_\tau vure_{\lambda_2}g_2$ (and so $S(\mathfrak{N})=Ae_\tau vute_{\lambda_1}g_1\oplus Ae_\sigma(ute_{\lambda_1}g_1-ure_{\lambda_2}g_2)$).

III–3·5: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2+Ae_{\lambda_3}g_3$, where $Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$ is the same as in III–3·4; that is, $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\tau vute_{\lambda_1}g_1$, $e_\tau vute_{\lambda_1}g_1=e_\tau vure_{\lambda_2}g_2$, and $Ae_{\lambda_3}g_3$ is a module of Type I such that $Ae_{\lambda_3}g_3\approx Ae_\rho we_{\lambda_1}g_1/Ne_\sigma swe_{\lambda_1}g_1$, i.e. $Ae_{\lambda_3}\approx Ae_\rho$, $S(Ae_{\lambda_3}g_3)=Ne_{\lambda_3}g_3=Ae_\sigma se_\rho\zeta e_{\lambda_3}g_3$ (ζ in A, but not in N) (and so $Ae_{\lambda_3}\not\approx Ae_{\lambda_1}$, $Ae_{\lambda_3}\not\approx Ae_{\lambda_2}$), and $Ae_{\lambda_3}g_3\frown(Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2)=Ae_\sigma s\zeta e_{\lambda_3}g_3$, $e_\sigma s\zeta e_{\lambda_3}g_3=e_\sigma ute_{\lambda_1}g_1-e_\sigma ure_{\lambda_2}g_2$ (and hence $S(\mathfrak{N})=Ae_\tau vute_{\lambda_1}g_1\oplus Ae_\sigma s\zeta e_{\lambda_3}g_3$).

III–4·1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type III such that $Ne_{\lambda_1}g_1=Ae_\kappa te_{\lambda_1}g_1+Ae_\rho we_{\lambda_1}g_1$, $S(Ae_{\lambda_1}g_1)=Ae_\kappa te_{\lambda_1}g_1\frown Ae_\rho we_{\lambda_1}g_1=Ae_\sigma ue_\kappa te_{\lambda_1}g_1$($u$ in N) and $e_\sigma ue_\kappa te_{\lambda_1}g_1=e_\sigma se_\rho we_{\lambda_1}g_1$ (s in N).

III–4·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is a module of Type III such that $Ne_{\lambda_1}g_1=Ae_\kappa te_{\lambda_1}g_1+Ae_\rho we_{\lambda_1}g_1$, $S(Ae_{\lambda_1}g_1)=Ae_\kappa te_{\lambda_1}g_1\frown Ae_\rho we_{\lambda_1}g_1=Ne_\kappa te_{\lambda_1}g_1=Ae_\sigma ue_\kappa te_{\lambda_1}g_1$($u$ in N) and $e_\sigma ue_\kappa te_{\lambda_1}g_1=e_\sigma se_\rho we_{\lambda_1}g_1$($s$ in N), and $Ae_{\lambda_2}g_2$ is a module of Type I such that there exists a maximal monomorphism ψ: $Ae_\kappa te_{\lambda_1}g_1\to Ne_{\lambda_2}g_2$, $\psi(e_\kappa te_{\lambda_1}g_1)=e_\kappa re_{\lambda_2}g_2$ (r in N)(and so $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$, $Ae_\sigma\not\approx Ae_\kappa$ and $S(Ae_{\lambda_2}g_2)=Ae_\sigma ure_{\lambda_2}g_2$), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\kappa te_{\lambda_1}g_1$, $e_\kappa te_{\lambda_1}g_1=e_\kappa re_{\lambda_2}g_2$(and hence $S(\mathfrak{N})=Ae_\sigma ute_{\lambda_1}g_1$).

III–4·3: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are respectively the same as in III–4·2, but $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\sigma ute_{\lambda_1}g_1$, $e_\sigma ute_{\lambda_1}g_1=e_\sigma ure_{\lambda_2}g_2$(and so $S(\mathfrak{N})=Ae_\sigma ute_{\lambda_1}g_1\oplus Ae_\kappa(te_{\lambda_1}g_1-re_{\lambda_2}g_2)$).

III–4·4: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2+Ae_{\lambda_3}g_3$, where $Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$ is the same as in III–4·3; that is, $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\sigma ute_{\lambda_1}g_1$, $e_\sigma ute_{\lambda_1}g_1=e_\sigma ure_{\lambda_2}g_2$, and $Ae_{\lambda_3}g_3$ is a non-simple module of Type I such that there exists a monomorphism: $Ae_{\lambda_3}g_3\to Ne_{\lambda_2}g_2/Ae_\sigma ure_{\lambda_2}g_2$ (and so r in N^2), $S(Ae_{\lambda_3}g_3)=Ae_\kappa pe_{\lambda_3}g_3$ (p in N), and $Ae_{\lambda_3}g_3\frown(Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2)=Ae_\kappa pe_{\lambda_3}g_3$, $e_\kappa pe_{\lambda_3}g_3=e_\kappa te_{\lambda_1}g_1-e_\kappa re_{\lambda_2}g_2$(and hence $S(\mathfrak{N})=Ae_\sigma ute_{\lambda_1}g_1\oplus Ae_\kappa pe_{\lambda_3}g_3$).

IV–1·1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type IV such that $Ae_{\lambda_1}g_1/N^2e_{\lambda_1}g_1$ is uni-serial, $Ne_{\lambda_1}g_1=Ae_\mu se_{\lambda_1}g_1$($s$ in N), $N^2e_{\lambda_1}g_1=Ae_\kappa te_\mu se_{\lambda_1}g_1+Ae_\rho we_\mu se_{\lambda_1}g_1$ (t, w in N), $Ae_\kappa tse_{\lambda_1}g_1\frown Ae_\rho wse_{\lambda_1}g_1=Ne_\kappa tse_{\lambda_1}g_1=Ae_\sigma u_1e_\kappa tse_{\lambda_1}g_1$($u_1$ in N), $S(Ae_{\lambda_1}g_1)=Ne_\rho wse_{\lambda_1}g_1=Ae_\sigma u_2e_\rho wse_{\lambda_1}g_1\oplus Ae_\tau ve_\rho wse_{\lambda_1}g_1$($u_2$, v in N)(and so $Ae_\sigma\not\approx Ae_\tau$, $Ae_\kappa\not\approx Ae_\sigma$, $Ae_\kappa\not\approx Ae_\tau$) and $e_\sigma u_1tse_{\lambda_1}g_1=e_\sigma u_2wse_{\lambda_1}g_1$.

IV–1·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in IV–1·1, and $Ae_{\lambda_2}g_2$ is a module of Type I such that $Ae_{\lambda_2}g_2\approx Ae_\rho wse_{\lambda_1}g_1/Ae_\sigma u_2wse_{\lambda_1}g_1$(and so $Ae_{\lambda_2}\approx Ae_\rho$, $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$), $S(Ae_{\lambda_2}g_2)=Ne_{\lambda_2}g_2=Ae_\tau ve_\rho\zeta e_{\lambda_2}g_2$($\zeta$ in A, but not in N), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\tau vwse_{\lambda_1}g_1$, $e_\tau vwse_{\lambda_1}g_1=e_\tau v\zeta e_{\lambda_2}g_2$

(and hence $S(\mathfrak{N})=Ae_\sigma u_2wse_{\lambda_1}g_1\oplus Ae_\tau vwse_{\lambda_1}g_1$).

IV–1·3: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in IV–1·1, and $Ae_{\lambda_2}g_2$ is a module of Type I such that $Ae_{\lambda_2}g_2\approx Ae_\mu se_{\lambda_1}g_1/Ae_\iota tse_{\lambda_1}g_1$ (and so $Ae_{\lambda_2}\approx Ae_\mu$, $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$), $Ne_{\lambda_2}g_2=Ae_\rho we_\mu\eta e_{\lambda_2}g_2$ (η in A, but not in N), $S(Ae_{\lambda_2}g_2)=N^2e_{\lambda_2}g_2=Ae_\tau vw\eta e_{\lambda_2}g_2$, and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\tau vwse_{\lambda_1}g_1$, $e_\tau vwse_{\lambda_1}g_1=e_\tau vw\eta e_{\lambda_2}g_2$ (and hence $S(\mathfrak{N})=Ae_\sigma u_2wse_{\lambda_1}g_1\oplus Ae_\tau vwse_{\lambda_1}g_1$).

IV–1·4: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in IV–1·1, and $Ae_{\lambda_2}g_2$ is a module of Type II such that $Ae_{\lambda_2}g_2\approx Ae_\mu se_{\lambda_1}g_1/Ae_\sigma u_1tse_{\lambda_1}g_1$ (and so $Ae_{\lambda_2}\approx Ae_\mu$, $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$), $Ne_{\lambda_2}g_2=Ae_\iota te_\mu\eta e_{\lambda_2}g_2\oplus Ae_\rho we_\mu\eta e_{\lambda_2}g_2$ (η in A, but not in N), $N^2e_{\lambda_2}g_2=Ae_\tau vw\eta e_{\lambda_2}g_2$ and $S(Ae_{\lambda_2}g_2)=Ae_\iota t\eta e_{\lambda_2}g_2\oplus Ae_\tau vw\eta e_{\lambda_2}g_2$, and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\tau vwse_{\lambda_1}g_1$, $e_\tau vwse_{\lambda_1}g_1=e_\tau vw\eta e_{\lambda_2}g_2$ (and hence $S(\mathfrak{N})$ $=Ae_\sigma u_2wse_{\lambda_1}g_1\oplus Ae_\tau vwse_{\lambda_1}g_1\oplus Ae_\iota t\eta e_{\lambda_2}g_2$).

IV–2·1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type IV such that $Ne_{\lambda_1}g_1=Ae_\iota te_{\lambda_1}g_1+Ae_\rho we_{\lambda_1}g_1$ (t, w in N), $Ae_\iota te_{\lambda_1}g_1\frown Ae_\rho we_{\lambda_1}g_1=Ae_\sigma u_1e_\iota te_{\lambda_1}g_1$ (u_1 in N), $Ne_\rho we_{\lambda_1}g_1=Ae_\sigma u_2e_\rho we_{\lambda_1}g_1\oplus Ae_\tau ve_\rho we_{\lambda_1}g_1$ (u_2, v in N), $e_\sigma u_1te_{\lambda_1}g_1=e_\sigma u_2we_{\lambda_1}g_1$ and $S(Ae_{\lambda_1}g_1)=Ae_\sigma u_2we_{\lambda_1}g_1\oplus Ae_\nu qe_\tau vwe_{\lambda_1}g_1$ (q in A) (and of course $Ae_\sigma\not\approx Ae_\nu$).

IV–2·2: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in IV–2·1, and $Ae_{\lambda_2}g_2$ is a module of Type I such that $Ae_{\lambda_2}g_2\approx Ae_\rho we_{\lambda_1}g_1/Ae_\sigma u_2we_{\lambda_1}g_1$ (and so $Ae_{\lambda_2}\approx Ae_\rho$, $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$), and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\tau vwe_{\lambda_1}g_1$, $e_\tau vwe_{\lambda_1}g_1=e_\tau ve_\rho\zeta e_{\lambda_2}g_2$ (ζ in A, but not in N) (and hence $S(\mathfrak{N})=Ae_\sigma u_2we_{\lambda_1}g_1\oplus Ae_\nu qvwe_{\lambda_1}g_1$).

IV–2·3: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_2}g_2$ are respectively the same as in IV–2·2, but $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2=Ae_\mu pe_\tau vwe_{\lambda_1}g_1$ (p in N)(and so q in N), $e_\mu pvwe_{\lambda_1}g_1=e_\mu pve_\rho\zeta e_{\lambda_2}g_2$ (ζ in A, but not in N)(and hence if we put $C(Ae_\mu pvwe_{\lambda_1}g_1)=Ae_\omega se_\tau vwe_{\lambda_1}g_1$ (s in A) and $C(Ae_\mu pv\zeta e_{\lambda_2}g_2)=Ae_\omega se_\tau v\zeta e_{\lambda_2}g_2$, then $S(\mathfrak{N})=Ae_\sigma u_2we_{\lambda_1}g_1\oplus Ae_\nu qvwe_{\lambda_1}g_1\oplus Ae_\omega(svwe_{\lambda_1}g_1-sv\zeta e_{\lambda_2}g_2)$ and $Ae_\omega\not\approx Ae_\sigma$, $Ae_\omega\not\approx Ae_\nu$).

IV–2·4: $\mathfrak{N}=Ae_{\lambda_1}g_1+Ae_{\lambda_2}g_2$, where $Ae_{\lambda_1}g_1$ is the same as in IV–2·1, and $Ae_{\lambda_2}g_2$ is a module of Type I such that there exists a maximal monomorphism ψ: $Ae_\rho we_{\lambda_1}g_1/Ae_\tau vwe_{\lambda_1}g_1\to Ne_{\lambda_2}g_2$, $\psi(e_\rho we_{\lambda_1}g_1)=e_\rho re_{\lambda_2}g_2$ (r in N) (and so $Ae_{\lambda_2}\not\approx Ae_{\lambda_1}$), and $S(Ae_{\lambda_2}g_2)=Ae_\sigma u_2re_{\lambda_2}g_2$, and $Ae_{\lambda_1}g_1\frown Ae_{\lambda_2}g_2$ $=Ae_\sigma u_2we_{\lambda_1}g_1$, $e_\sigma u_2we_{\lambda_1}g_1=e_\sigma u_2re_{\lambda_2}g_2$(and hence $S(\mathfrak{N})=Ae_\sigma u_2we_{\lambda_1}g_1\oplus Ae_\nu qvwe_{\lambda_1}g_1$).

V–1: $\mathfrak{N}=Ae_{\lambda_1}g_1$, where $Ae_{\lambda_1}g_1$ is a module of Type V.

The details of the proof of our results stated in Parts I and II will be published elsewhere.