1. On the Measure-Bend of Parametric Curves

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1. Curves straightenable on a set. In the present continuation of our recent note [5] we shall derive some further measure-theoretic properties of parametric curves. Throughout the note the space $\boldsymbol{R}^m$ will be assumed at least 2-dimensional, while all the curves considered will be defined over $\boldsymbol{R}$ (unless stated to the contrary) and situated in $\boldsymbol{R}^m$. A curve $\varphi(t)$ will be termed *straightenable* (or *of bounded bend*) on a set E of real numbers iff the bend $\Omega(\varphi; E)$ is finite, and *locally straightenable* (or *of locally bounded bend*) iff φ is straightenable on all linear closed intervals. Let us begin our argument with a lemma which extends [1]§64.

LEMMA. *If a curve φ is straightenable on a set E as well as bounded on E, it is rectifiable on the same set. In consequence, a locally straightenable curve is locally rectifiable whenever it is locally bounded.*

REMARK. Simple examples show that the boundedness of φ on E is essential for the validity of the assertion (cf. the remark of [1]§64).

PROOF. By change of parameter if necessary, we may suppose without loss of generality that E is a bounded set. Let I_0 denote generically an open interval. We shall show in the first place that if $\Omega(\varphi; I_0E)<\pi/3$, the curve φ is rectifiable on I_0E and we have $L(\varphi; I_0E)\leqq 2\mathrm{d}(\varphi[I_0E])$, where for any set X in $\boldsymbol{R}^m$ we denote by $\mathrm{d}(X)$ the diameter of X. For this purpose we may suppose $L(I_0E)$ positive. It suffices to derive $L(IE)\leqq 2|\varphi(I)|$ for each closed interval I contained in I_0 and whose endpoints belong to E. For it is obvious, by definition of length, that $L(I_0E)$ is the supremum of $L(IE)$. We now distinguish two cases according as the increment $\varphi(I)$ vanishes or not. If $\varphi(I)=0$, then φ must be constant on the set IE and hence $L(IE)=0=2|\varphi(I)|$; indeed we should otherwise get the evident contradiction $\Omega(I_0E)\geqq\Omega(IE)\geqq\pi$. If on the other hand $\varphi(I)\neq 0$, then $L(IE)\leqq 2|\varphi(I)|$ follows easily by an argument similar to that of [1]§63. We leave the details to the reader.

Writing $\theta=\Omega(I_0E)$ for an arbitrary $I_0=(a, b)$, we shall further show that there exists in I_0 a point c such that $\Omega((a, c)\cdot E)\leqq\theta/2$ and $\Omega((c, b)\cdot E)\leqq\theta/2$. Of course we need only consider the case $\theta>0$. It is clear that (i) the supremum of the bend $\Omega(JE)$, where J ranges over the closed intervals in I_0, coincides with $\Omega(I_0E)=\theta$ and that (ii) if $I_1,\cdots,I_n$ are any disjoint finite sequence of intervals in I_0, then $\Omega(I_1E)+\cdots+\Omega(I_nE)\leqq\theta$ (see the lemma of [5] § 4). Consider now the set A of the points t of I_0 such that $\Omega((a,t)\cdot E)\leqq\theta/2$. It follows at once from (i) and (ii) that all the points of I_0 sufficiently near the point a certainly belong to A. So that, if we put $c=\sup A$, then $a<c\leqq b$. But we cannot have $c=b$ here, since otherwise (i) would imply the absurd relation $\theta=\Omega(I_0E)\leqq\theta/2$. Moreover, replacing in (i) the interval I_0 by (a,c), we find at once that $\Omega((a,c)\cdot E)\leqq\theta/2$. It remains to examine the inequality $\Omega((c,b)\cdot E)\leqq\theta/2$. If this were false, then the statement (i), where we replace I_0 by (c,b), would imply the existence of a point c' subject to the conditions $c<c'<b$ and $\Omega((c',b)\cdot E)>\theta/2$. The last inequality, combined with (ii), would give us $\Omega((a,c')\cdot E)<\theta/2$, which evidently contradicts the definition of the point c.

Hitherto the open interval I_0 has been variable. Let us now fix it so as to comprise the set E (which is bounded by assumption), and let us choose the point c whose existence has just been established. The procedure that replaces I_0 by the pair of intervals (a,c) and (c,b) will now be repeated for each of (a,c) and (c,b) separately; and so on we proceed until we obtain in I_0 a finite disjoint sequence of open intervals $K_1,\cdots,K_p$ such that $I_0-(K_1\smile\cdots\smile K_p)$ is a finite set and such that $\Omega(K_iE)<\pi/3$ for $i=1,\cdots,p$. Then we must have $L(K_iE)\leqq 2\mathrm{d}(\varphi[K_iE])\leqq 2\mathrm{d}(\varphi[E])$ for each i by what has already been proved at the beginning. The assertion follows now from

$$L(E)\leqq L(K_1E)+\cdots+L(K_pE)+2p\,\mathrm{d}(\varphi[E])\leqq 4p\,\mathrm{d}(\varphi[E])<+\infty.$$

THEOREM. *If a curve $\varphi(t)$ is straightenable on a set E as well as bounded on E, there exists a bounded, rectifiable, straightenable curve $\psi(t)$ coinciding on E with $\varphi(t)$ and satisfying the relations*

$$L(\psi;\boldsymbol{R})=L(\varphi;E) \text{ and } \Omega(\psi;\boldsymbol{R})=\Omega(\varphi;E).$$

PROOF. It follows from the foregoing lemma that φ is rectifiable on E. Assuming E nonvoid as we may, consider the curve υ which is defined on the closure $\overline{E}$ of E, by means of the curve φ, in precisely the same way as at the beginning of the proof for the lemma of [5] § 3. Then it is readily found that $L(\upsilon;\overline{E})=L(\varphi;E)$ and $\Omega(\upsilon;\overline{E})=\Omega(\varphi;E)$. We may thus suppose from the first that E is a nonvoid closed set (other than $\boldsymbol{R}$).

To determine a curve ψ conforming to the assertion, we put in the first place $\psi(t)=\varphi(t)$ for each $t\in E$. Let now I be any interval contiguous to E. We extend to I the definition of $\psi(t)$ by requiring it to be linear on I when I is a finite interval, and to be constant on I when I is infinite. We see at once that the curve ψ, thus determined uniquely on the whole real line, fulfils the requirements of the theorem.

2. Extension of a previous result. We gave in [3]§5 a sufficient condition for the measure-length of a spheric curve to coincide on a given set with the spheric measure-length of the same curve. Now we have to extend that result to the following form, the proof of which will be found even simpler than before.

LEMMA. *If γ is a spheric curve, then $L_*(\gamma; X)=\Lambda_*(\gamma; X)$ for each set X at whose points the curve is continuous.*

PROOF. Suppose X nonvoid and consider an arbitrary real number $\alpha>1$. By hypothesis it is possible to enclose each point t of X in an open interval $U(t)$ such that $\gamma(p)\diamond\gamma(q)\leqq\alpha|\gamma(p)-\gamma(q)|$ whenever p and q are a pair of points of $U(t)$. Let us denote by D the join of all the intervals $U(t)$. Then $\Lambda(J)\leqq\alpha L(J)$ whenever J is a closed interval in D. To see this, we need merely observe that if J is fixed and ε is a sufficiently small positive number, then for each closed interval $K=[p, q]$ contained in J and with length $<\varepsilon$ there exists in the set X a point t for which $K\subset U(t)$, so that $\gamma(p)\diamond\gamma(q)\leqq\alpha|\gamma(K)|$ by definition of $U(t)$. The proof of this is immediate by *reductio ad absurdum*.

The result just obtained implies that $\Lambda(I)\leqq\alpha L(I)$ for each endless interval $I\subset D$; indeed $\Lambda(I)$ is clearly the supremum of $\Lambda(J)$ for all closed intervals $J\subset I$ and similarly for $L(I)$. If, therefore, we cover X by any sequence Δ of endless intervals lying in D, we must have $\Lambda_*(X)\leqq\Lambda(\Delta)\leqq\alpha L(\Delta)$. This yields us $\Lambda_*(X)\leqq\alpha L_*(X)$, since $L_*(X)$ is easily seen to be the infimum of $L(\Delta)$. Making $\alpha\to 1$, we obtain at once $\Lambda_*(X)\leqq L_*(X)$. The converse inequality being obvious, the proof is complete.

3. Measure-bend of general curves. Our definition of measure-bend, given in [2]§3, was confined to the case where the curve under consideration is light. As is immediately seen, however, the same definition applies as well to any parametric curve $\varphi(t)$ whatsoever. From now on the term *measure-bend* of φ on a set E will be interpreted in this extended sense and we shall use for it the notation $\Omega_*(\varphi; E)$ as before. The object of this section is to prove a useful theorem concerning measure-bend. It will be applied in our forthcoming continuation of the present note.

We easily verify that $\Omega_*(\varphi; E)$, *considered as function of E, is always an outer measure of Carathéodory.* But it should be noted that the relation $\Omega_*(\varphi; I)=\Omega(\varphi; I)$, which was proved in [2]§3 for endless intervals I when the curve φ is light, holds no more when φ is a general curve. This may be ascertained by simple examples.

LEMMA. *Given a light curve φ, let γ be a direction curve of φ,* *i.e. let $\gamma(t)$ be a derived direction of φ at each point t of* $\boldsymbol{R}$ (see [1] § 44). *Then $\Lambda_*(\gamma; E) \leqq \Omega_*(\varphi; E)$ for any set E. Moreover, the sign of equality holds in this inequality provided, in addition, that φ is continuous and γ is a unilateral direction curve of φ* (see [1] § 73).

PROOF. The first half of the assertion follows directly from the definitions of Λ_* and Ω_* in virtue of the proposition of [1] § 45. To deduce the second half, suppose E nonvoid and cover E in any manner by a sequence Δ of endless intervals. For each interval I of Δ we then find, by the fundamental theorem of bend theory (*vide* [1] § 95), that $\Omega(\varphi; I) = \Lambda(\gamma; I)$. Summing this over all I, we obtain $\Omega_*(\varphi; E) \leqq \Omega(\varphi; \Delta) = \Lambda(\gamma; \Delta)$, whence we derive $\Omega_*(\varphi; E) \leqq \Lambda_*(\gamma; E)$ since Δ is arbitrary. The converse inequality being already established, the proof is complete.

THEOREM. *Given a light continuous curve $\varphi(t)$ and a set E of real numbers, suppose that $\Omega_*(\varphi; M) = 0$ for every countable set $M \subset E$.* (i) *Then $\Omega_*(\varphi; E) \leqq L(\gamma; E)$ whenever γ is a unilateral direction curve (on $\boldsymbol{R}$) for the curve φ, and* (ii) *we have $\Omega_*(\varphi; E) \leqq \Omega_*(\psi; E)$ for each curve ψ which coincides on E with φ.*

REMARK. Since the measure-bend of a curve is always an outer Carathéodory measure, the condition $\Omega_*(\varphi; M) = 0$ of the theorem is equivalent to the seemingly weaker hypothesis that $\Omega_*(\varphi; \{t\}) = 0$ for every point t of E.

PROOF. We may suppose E nonvoid in both parts of the assertion.

re (i): By the above lemma we have $\Lambda_*(\gamma; X) = \Omega_*(\varphi; X)$ for every set X and so $\Lambda_*(\gamma; \{t\})$ vanishes whenever $t \in E$. This implies continuity of γ at all points of E, and consequently we find in view of the lemma of § 2 that $\Omega_*(\varphi; E) = \Lambda_*(\gamma; E) = L_*(\gamma; E)$. Statement (i) is therefore reduced to the inequality $L_*(\gamma; E) \leqq L(\gamma; E)$, which may be proved as follows.

Each point t of E can be enclosed, on account of $\Lambda_*(\gamma; \{t\}) = 0$ mentioned above, in an open interval $U(t)$ with rational extremities and for which $L(\gamma; U(t)) < 1$. So that there exists in E an infinite sequence of points $t_1, t_2, \cdots$ such that E is already covered by the sequence $U(t_1), U(t_2), \cdots$. If, therefore, we write for brevity $U_n = U(t_1) \smile \cdots \smile U(t_n)$ where $n = 1, 2, \cdots$, then $L_*(\gamma; U_n) < n$ for each n and further it follows from Theorem (4.6) on p. 46 of Saks [6] that $L_*(\gamma; EU_n) \rightarrow L_*(\gamma; E)$ as $n \rightarrow +\infty$. It is thus enough to ascertain $L_*(\gamma; EU_n) \leqq L(\gamma; E)$ for each fixed n.

Now the set U_n, which is the join of a finite number of open intervals, can be decomposed into a finite disjoint sequence Δ of open intervals I, where we observe that $L(\gamma; I) = L_*(\gamma; I) \leqq L_*(\gamma; U_n) < n$. We then have $L_*(\gamma; EU_n) = L_*(\gamma; E\Delta)$, using again Theorem (4.6) just quoted. Since evidently $L(\gamma; E\Delta) \leqq L(\gamma; E)$ on the other hand, it only remains to show that $L_*(\gamma; EI) \leqq L(\gamma; EI)$ for each I in $\varDelta$. By change of parameter, however, this relation follows directly from the theorem of [4] § 4.

re (ii): Let E_0 be the set of all the right-hand points of condensation for E and write $E_1 = E - E_0$. In other words, a point t belongs to E_0 iff every closed interval whose left-hand extremity is t contains an uncountable infinity of points of E. Then E_1 is countable on account of a lemma to be proved in the next section, and so every point of E_0 is a right-hand point of condensation for EE_0. Moreover $\varOmega_*(\varphi; E_1)$ vanishes by hypothesis, so that we must have $\varOmega_*(\varphi; E) = \varOmega_*(\varphi; EE_0)$. It follows that we may assume $E_0 \supset E$ from now on.

This being so, let x be a fixed point of E. Then, since $\varOmega_*(\varphi; \{x\})$ vanishes, we find that $\varOmega(\varphi; K) < 1$ for every sufficiently short closed interval K for which x is the left-hand extremity. By our assumption $E_0 \supset E$ such an interval K contains an infinity of points $x' > x$ of E and, on account of the lightness of φ, every x' fulfils the condition $\varphi(x') \neq \varphi(x)$; for otherwise we should get at once the contradiction $\varOmega(\varphi; K) \geqq \pi$ in virtue of the proposition of [1] § 60. It follows that there is in the set E a strictly decreasing sequence of points $x_1, x_2, \cdots$ tending to x and such that $\varphi(x_n) \neq \varphi(x)$ for each n and moreover, as $n \to +\infty$, the direction of the vector $\varphi(x_n) - \varphi(x)$ tends to a unit-vector $\gamma_0(x)$. We have thus constructed on E a right-hand direction curve γ_0 for the curve φ. Since φ is light, we can further extend the definition of γ_0 to the whole real line in such a manner that the resulting curve, which we shall denote by γ, is now a right-hand direction curve on $\boldsymbol{R}$ for φ.

So far we have only been concerned whith the curve φ. Let us turn now to the consideration of ψ. By definition of measure-bend, $\varOmega_*(\psi; E)$ is the infimum of $\varOmega(\psi; \varTheta)$ where $\varTheta$ is an arbitrary sequence of endless intervals which together cover E. On the other hand the curve γ_0 defined just now is evidently a right-hand direction curve over E for ψ and we therefore find by [1] § 45 that $\varOmega(\psi; J) \geqq L(\gamma_0; EJ) = L(\gamma; EJ)$ for each interval J in $\varTheta$. But, by statement (i) established already, we have $L(\gamma; E) \geqq \varOmega_*(\varphi; E)$, where the set E may clearly be replaced by any of its subsets; so that in particular $L(\gamma; EJ) \geqq \varOmega_*(\varphi; EJ)$ for each J. It follows that $\varOmega(\psi; J) \geqq \varOmega_*(\varphi; EJ)$, which leads on summation to $\varOmega(\psi; \varTheta) \geqq \varOmega_*(\varphi; E\varTheta) \geqq \varOmega_*(\varphi; E)$. Taking the infimum of $\varOmega(\psi; \varTheta)$ for all choices of $\varTheta$ we get finally $\varOmega_*(\psi; E) \geqq \varOmega_*(\varphi; E)$, completing the proof.

4. A lemma from point set theory. Given a set S of real numbers, we shall say for the moment that an interval I, which may be of any type, is *sparse* (with respect to S), iff the intersection SI is at most a countable set. Such an interval will further be called *maximal* iff it is contained in no other sparse interval. We shall enumerate in the following lines some simple properties of sparse intervals. The proofs are immediate and left to the reader. (i) An interval is sparse when, and only when, every closed interval contained in it is sparse; (ii) the join of any intersecting pair of sparse intervals is again a sparse interval; (iii) no pair of maximal sparse intervals can intersect unless they coincide completely; (iv) there exists at most a countable infinity of maximal sparse intervals; (v) each sparse interval is contained in some maximal sparse interval.

We now conclude the present note with the following result which is a direct consequence of the avove statements.

LEMMA. *A subset T of S is countable whenever each point of T can be enclosed in an interval sparse with respect to S.*

ADDED IN PROOF. With the aid of the theorem of §3 we shall derive in our forthcoming note the following result which resembles the theorem of [4]§4 and involves bend, measure-bend, and reduced measure-bend:

Given a curve φ and a set E, suppose that $\Omega_*(\varphi; X)=0$ for every countable set $X\subset E$. Then $\Upsilon(\varphi; E)=\Omega_*(\varphi; E)\leqq\Omega(\varphi; E)$.

References

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