25. *Representations of Compact Groups Realized by Spherical Functions on Symmetric Spaces*

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1. The problem of determining the irreducible representations of a connected compact semisimple Lie group G realized by spherical functions on a symmetric Riemannian space G/K was first treated by E. Cartan [1]. In the present note we shall give a more explicit determination of these representations by means of "Satake diagrams". Our theory could be founded on the basis of the fundamental result of Cartan ([1], p. 241). It should be noticed however that, although this result is valid, its proof in [1] was not complete. So we shall start anew from the beginning. The detailed discussion with proofs will appear elsewhere.

2. Let G be a compact group and K be a closed subgroup of G. The totality $C(G/K)$ of complex valued continuous functions on G/K becomes the representation space of the representation $(T, C(G/K))$ of G if we define $T_g f = f \circ g^{-1}$, $f \in C(G/K)$. An element of an irreducible invariant subspace of $C(G/K)$ under this representation is called a spherical function on G/K. A representation (ρ, V) of G is called a representation realized by spherical functions on G/K if (ρ, V) is equivalent to one which is an irreducible component of $(T, C(G/K))$. It is easily seen that an irreducible representation (ρ, V) of G is realized by spherical functions on G/K if and only if $\rho(K)$ has a non-zero invariant in V (cf. E. Cartan [1]).

The most interesting case, to which we confine ourselves, is when G is a Lie group and G/K is a symmetric Riemannian space. In this case spherical functions are characterized as the simultaneous eigenfunctions of all invariant linear differential operators on G/K (cf. M. Sugiura [3]).

3. Let σ be an involutive automorphism of a connected compact semisimple Lie group G and K be the totality of fixed points under σ. K is a closed subgroup of G and the coset space G/K has the structure of a symmetric Riemannian space. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. For any subspace V of $\mathfrak{g}$, we denote by $V^{\perp}$ the orthogonal complement of V with respect to the Killing form $(X, Y) = Tr\ adXadY$ which is negative definite on $\mathfrak{g}$.

Put $\mathfrak{p} = \mathfrak{k}^{\perp}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$, and $\mathfrak{h}$ be a maximal abelian subalgebra in $\mathfrak{g}$ containing $\mathfrak{a}$. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The complexification $\mathfrak{h}^c$ of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^c$ (the complexification of $\mathfrak{g}$).

Put $\mathfrak{b}=\mathfrak{a}^{\perp}\frown\mathfrak{h}$, then $\mathfrak{h}=\mathfrak{a}+\mathfrak{b}$ (direct sum). We choose a basis $X_1,\cdots,X_l$ of $\mathfrak{h}$ so that $X_1,\cdots,X_p$ forms a basis of $\mathfrak{a}$ and $X_{p+1},\cdots,X_l$ is a basis of $\mathfrak{b}$. We introduce a linear order in $\sqrt{-1}\,\mathfrak{h}$ using the basis $\sqrt{-1}\,X_1,\cdots,\sqrt{-1}\,X_l$. A real valued linear form λ on $\sqrt{-1}\,\mathfrak{h}$ is identified with an element H_λ of $\sqrt{-1}\,\mathfrak{h}$ satisfying $\lambda(H)=(H,H_\lambda)$ for all H in $\sqrt{-1}\,\mathfrak{h}$. So we can speak of the order and the inner product between weights and roots of $\mathfrak{g}$. If the restriction α to $\mathfrak{a}$ of a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is not equal to zero, then α is called a root of the symmetric space G/K or of the symmetric pair $(\mathfrak{g},\mathfrak{k})$.

4. We have the following theorems.

Theorem 1. Let (ρ, V) be an irreducible representation of G realized by spherical functions on G/K and λ be the highest weight of ρ. Then λ satisfies the following two conditions 1) and 2).

1) $\lambda(\mathfrak{b})=0$.

2) $2(\lambda,\alpha)/(\alpha,\alpha)$ is an even integers for every root of G/K.

Theorem 2. Let $\dim\mathfrak{a}=p$, then there exist exactly p dominant integral forms $\mu_1,\cdots,\mu_p$ on $\mathfrak{h}^c$ such that the totality of the dominant integral forms on $\mathfrak{h}^c$ satisfying 1) and 2) in Theorem 1 is $\left\{\sum_{i=1}^{p} m_i\mu_i;\ m_i\in\boldsymbol{Z}, m_i\geqq 0\right\}$.

5. To determine these μ_i's, we use the "Satake diagram", which we shall define as follows (cf. [2]).

Let $\Delta=\{\alpha_1,\cdots,\alpha_l\}$ be the totality of the simple roots with respect to the above defined order in $\sqrt{-1}\,\mathfrak{h}$. We denote by τ the restriction to $\mathfrak{h}^c$ of the conjugation of $\mathfrak{g}^c$ with respect to the real form $\mathfrak{g}_0=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$. τ is uniquely expressed as

$$\tau=SP, \tag{1}$$

where S is an element of the Weyl group W and P transforms Δ onto itself. Now, the Satake diagram (of $(\mathfrak{g},\mathfrak{k})$, or of $\mathfrak{g}_0$) is the Dynkin diagram of $\mathfrak{g}$ with the following two additional properties a) and b).

a) A simple root lying not in $\sqrt{-1}\,\mathfrak{b}$ is represented by a white vertex and a simple root in $\sqrt{-1}\,\mathfrak{b}$ is represented by a black vertex.

b) Two simple root α_i and α_j are connected by an arrow $\curvearrowright$ if the transformation P in (1) transforms α_i to α_j. The Satake diagrams for classical simple groups can be seen in I. Satake [2]. Those for exceptional simple groups are listed at the end of this note.

Theorem 3. Let $\lambda_1,\cdots,\lambda_l$ be the dominant integral forms on $\mathfrak{h}^c$ satisfying

$$2(\lambda_i,\alpha_j)/(\alpha_j,\alpha_j)=\delta_{ij},$$

where $\Delta=\{\alpha_1,\cdots,\alpha_l\}$ is the totality of simple roots of $\mathfrak{g}^c$ with respect to the above defined order. Let $\alpha_i\notin\sqrt{-1}\,\mathfrak{b}$, $1\leqq i\leqq l-l_0$; $\alpha_i\in\sqrt{-1}\,\mathfrak{b}$,

$l-l_0+1 \leqq i \leqq 1$ and suppose that $P\alpha_i=\alpha_i$; $1 \leqq i \leqq p_1$ and $P\alpha_i=\alpha_{i+p_2}$, $p_1+1 \leqq i \leqq p_1+p_2=p$. (Notice $l-l_0=p_1+2p_2$, cf. [2].)

Then the p dominant integral forms $\mu_1, \cdots, \mu_p$ in Theorem 2 are determined from the Satake diagram of G/K as follows:

$$\mu_i=\begin{cases} \lambda_i, & 1 \leqq i \leqq p_1 \text{ and } i \text{ is connected with a black vertex,} \\ 2\lambda_i, & 1 \leqq i \leqq p_1 \text{ and } i \text{ is not connected with a black vertex,} \\ \lambda_i+\lambda_{i+p_2}, & p_1+1 \leqq i \leqq p_1+p_2. \end{cases}$$

Theorem 4. Assume that G is simply connected. Then the converse of Theorem 1 is valid, i.e., every irreducible representation of G with the highest weight λ satisfying the conditions 1) and 2) in Theorem 1 is realized by spherical function on G/K.

Theorem 1, 2, 3, and 4 determine completely the considered representations.

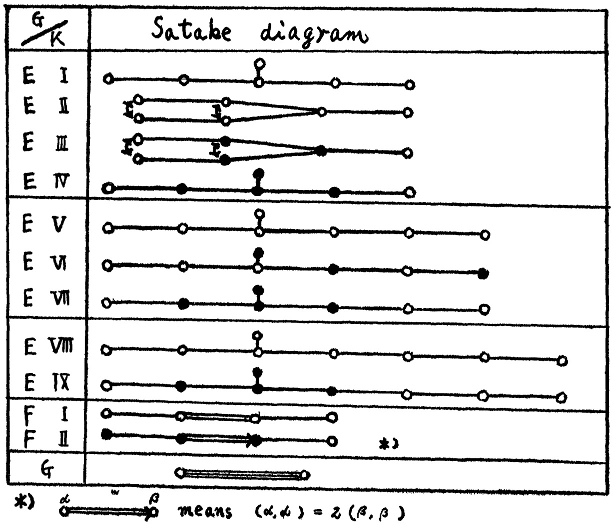

Fig. 1

References

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