21. *Multipliers of Banach Algebras*

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1. For a commutative semi-simple Banach algebra A, regarding as an algebra of continuous functions on the space X of all maximal regular ideals of A, a multiplier g of A is introduced by Helgason [2] as a function on X satisfying

$$gA\subseteq A. \tag{1}$$

The notion of multiplier is recently generalized by Wang [3] when A is a commutative Banach algebra "without order" in the sense that $aA=0$ implies $a=0$: A map g of A into A is called a multiplier if g satisfies

$$(ga)ba=(gb), \tag{2}$$

or equivalently,

$$g(ab)=(ga)b, \tag{3}$$

for any a and b in A. A similar observation is also given by Foias [1] who used "factor function" instead of multiplier and rather restrictive conditions on A. The definition is equivalent to that of Helgason if A is semi-simple. Foias and Wang proved, among others, the following

THEOREM 1. *The multiplier algebra $M(A)$, the set of all multipliers of A, is a Banach algebra having the identity and closed with respect to the strong operator topology.*

In non-commutative case, (2) is not equivalent to (3). However, it is reasonable to expect that a linear operator g defined by (3) plays some role even in non-commutative case, since an endomorphism g satisfying (3) is known as an admissible A-endomorphism in the theory of classical rings. In the below, it will be shown that Theorem 1 is also true for a non-commutative Banach algebra.

2. A (*left*) *multiplier* of a (not necessarily commutative) Banach algebra A is a (bounded) linear operator g which maps A into A satisfying (3). By this definition, it is obvious that $M(A)$ forms a normed algebra with the identity by the operator norm, since $g,f\in M(A)$ implies

$$fg(ab)=f[(ga)b]=(fga)b.$$

If g_α converges strongly to a linear operator g, then

$$g(ab)=\lim_\alpha g_\alpha(ab)=\lim_\alpha(g_\alpha a)b=(ga)b$$

shows that g belongs to $M(A)$, whence $M(A)$ is strongly closed. Naturally, $M(A)$ is complete with respect to the operator norm.

This completes the proof of Theorem 1.

A *right weak approximate identity* $\{e_\alpha\}$ of a Banach algebra A is a directed family of elements of A for which $e_\alpha a$ converges to a for every $a \in A$. The following theorem is also a non-commutative version of a theorem of Foias-Wang:

THEOREM 2. *$M(A)$ contains A as a left ideal. If A has a right weak approximate identity, then A is dense in $M(A)$ with respect to the strong operator topology. Moreover, if A has the identity, then A coincides with $M(A)$.*

Since every element a of A satisfies (3), A is a subalgebra of $M(A)$. For any $g \in M(A)$, ga is an element of A, whence A is a left ideal. If $e_\alpha a$ converges to a, then $ge_\alpha a$ converges to ga for every $a \in A$, whence ge_α converges to g strongly, which shows that A is strongly dense in $M(A)$. Finally, if A has the identity, then $A = M(A)$ since A is a left ideal by the above.

3. Similar theorems are also true for *right multipliers* defined by

$$(ab)h = a(bh). \tag{4}$$

Changing (4) into an operator form, a right multiplier T of a Banach algebra A satisfies

$$T(ab) = a(Tb), \tag{5}$$

which is already discussed by Wendel [4] as a *left centralizer* when A is the group algebra $L(G)$ of a locally compact group G. Since Wendel proved that the left centralizers and the Radon measures on G are corresponded isomorphically by $Ta = a * \mu$, the following non-commutative generalization of a theorem of Foias-Wang is now obvious:

THEOREM 3. *The right multiplier algebra of the group algebra of a locally compact group is isomorphic to the algebra of all Radon measures on the group.*

References

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