35. Decomposition of Kronecker Products of Representations of the Inhomogeneous Lorentz Group

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We shall consider irreducible decompositon of the Kronecker products of unitary representations of the proper orthochronous inhomogeneous Lorentz group which we shall denote by G. In the present paper we give explicit solution of this problem, except some cases. Our method is based on the theory of induced representation, which was considered by G. W. Mackey [1]. The details of the results will be reported in another paper.

§ 1. It is known that all irreducible unitary representations of G are classified in the following types.

a) $\mathfrak{D}^0(\sigma)$. Let H be the homogeneous Lorentz group. We can consider it the factor group of G by the real four-dimensional vector subgroup N of G. We can define canonically an irreducible representation $\mathfrak{D}^0(\sigma)$ of G from an irreducible representation σ of the factor group H. σ is characterized by a pair of parameters (s, t), according whose values σ's are classified into three series. (i) Principal series: s is a non-negative integer, t is pure imaginary. (ii) Supplementary series: $s=0$, $0<t<1$. (iii) Identity representation: $(s,t)=(0,1)$.

b) $\mathfrak{D}^1(\rho, b)$. Consider the three-dimensional rotation group R, which is a subgroup of H. Its irreducible representation $\rho\equiv\rho(m)$ is decided by the highest weight m, which is a non-negative integer. We choose a character $\chi(x)=\exp(ibx_1)$ $(b\neq 0)$ for the element $x=(x_1, x_2, x_3, x_0)$ of N, and construct a representation of RN as the product of $\rho(m)$ and $\chi(x)$, and lastly we induce a unitary representation $\mathfrak{D}^1(\rho, b)$ of G from this representation of RN. Then $\mathfrak{D}^1(\rho, b)$ is irreducible.

c) $\mathfrak{D}^2(\lambda, c)$. In the case b), we replace R with the three-dimensional proper Lorentz group L, the representation ρ of R with an irreducible representation λ of L and the character $\exp(ibx_1)$ of N with $\exp(icx_0)$ $(c>0)$ respectively. Then the induced representation $\mathfrak{D}^2(\lambda, c)$ from $\exp(icx_0)\lambda$ of LN is irreducible. There are four series of irreducible representations λ of L: (i) Principal series: λ^l $(1/4\leq l)$. (ii) Supplementary series: λ^l $(0<l<1/4)$. (iii) Discrete series: λ_p $(p=\pm 1, \pm 2, \cdots)$. (iv) Identity representation: I.

d) $\mathfrak{D}^3(\kappa, +)$ and $\mathfrak{D}^3(\kappa, -)$. We employ the motion group M over two-dimensional Euclidean space as the third subgroup of G, and let κ

be an irreducible representation of M. Let $\mathfrak{D}^3(\kappa, +)$ (resp. $\mathfrak{D}^3(\kappa, -)$) be the induced representation of G from the representation $\exp(i(x_1+x_0))\kappa$ (resp. $\exp(-i(x_1+x_0))\kappa$) of MN. κ belongs one of following types.

i) κ_m(m: integer). A character $\exp(im\theta)$ of the rotational subgroup of M defines canonically an one-dimensional representation of M, which we denote by κ_m. ii) $\kappa_\rho(0<\rho<\infty)$. Let $\exp(i\rho r_1)$ be a character of the subgroup of translations in M which are parametrized by (r_1, r_2), and κ_ρ be the induced representation of M from $\exp(i\rho r_1)$.

§2. In virtue of the symmetry of Kronecker product, the representation $\mathfrak{R}_1\otimes\mathfrak{R}_2$ is unitary equivalent to $\mathfrak{R}_2\otimes\mathfrak{R}_1$ for two representations $\mathfrak{R}_1, \mathfrak{R}_2$ of G, so it is sufficient to consider only one of these two types of product. Under this consideration, our results are as follows.

I) $\mathfrak{D}^0(\sigma)\otimes\mathfrak{D}^0(\tau)$. In this case the representation to be decomposed is nothing else the representation of G defined canonically by the representation $\sigma\otimes\tau$ of H, and its decomposition is essentially equivalent to decompose Kronecker product of representations of the homogeneous Lorentz group H. This problem is solved completely by M. A. Naimark in the series of his works [2].

II) $\mathfrak{D}^0(\sigma(s,t))\otimes\mathfrak{D}^1(\rho(m),b)=\sum_n[w(s,t,m,n)]\mathfrak{D}^1(\rho(n),b)$, where $w(s,t,m,n)$ is the function defined as follows, and the brackets [ ] means the multiplicity of the succeeding component.
For $(s,t)\neq(0,1)$, $w(s,t,m,n)=2m+1$, (if $n\geq m+s$); $=m-s+n+1$, (if $m+s\geq n\geq|m-s|$); $=2s+1$, (if $m-s\geq n\geq 0$); $=0$, (otherwise).
And $w(0,1,m,n)=\delta_m^n$.

III) $\mathfrak{R}=\mathfrak{D}^0(\sigma)\otimes\mathfrak{D}^2(\lambda,c)$.

i) When σ is a representation of the principal series, we have the following formula:

$$\mathfrak{R}=\sum_{q\neq 0}[w(\lambda,s,q)]\mathfrak{D}^2(\lambda_q,c)\oplus[v(\lambda)]\int_{1/4}^{\infty}\mathfrak{D}^2(\lambda^l,c)\,dl.$$

where $w(\lambda^l,s,q)=\infty$.
$w(\lambda_p,s,q)=\infty$, (if $pq>0$); $=s-|p|-|q|+1$, (if $pq<0$, and $s-|p|\geq|q|>0$); $=0$, (otherwise).
$w(I,s,q)=1$, (if $s\geq|q|>0$); $=0$, (otherwise).
$v(\lambda)=2$, (if $\lambda=I$); $=\infty$, (otherwise).

ii) When σ is the identity representation, then

$$\mathfrak{R}=I\otimes\mathfrak{D}^2(\lambda,c)\cong\mathfrak{D}^2(\lambda,c).$$

iii) When σ is a representation of the supplementary series, the problem is unsolved.

IV) $\mathfrak{R}_+=\mathfrak{D}^0(\sigma)\otimes\mathfrak{D}^3(\kappa,+)$ and $\mathfrak{R}_-=\mathfrak{D}^0(\sigma)\otimes\mathfrak{D}^3(\kappa,-)$.
If $\sigma=I$, it is trivial. Hence let $\sigma\neq I$. Then

$$\mathfrak{R}_+\cong\int_0^\infty\mathfrak{D}^3(\kappa^\rho,+)d\rho,\qquad \mathfrak{R}_-\cong\int_0^\infty\mathfrak{D}^3(\kappa^\rho,-)d\rho,\ (\text{if }\kappa=\kappa_m).$$

$$\mathfrak{R}_+\cong[\infty]\int_0^\infty \mathfrak{D}^3(\kappa^\rho,+)d\rho,\quad \mathfrak{R}_-\cong[\infty]\int_0^\infty \mathfrak{D}^3(\kappa^\rho,-)d\rho,\ (\text{if } \kappa=\kappa^\rho).$$

For cases V)—IX) of the remaining cases the results take similar forms, which are expressed in the following general formula:

$$\mathfrak{R}=\sum_{h\geq 0}[u]\int_{\alpha(1)}^{\alpha(2)}\mathfrak{D}^1(\rho(h),s)ds\oplus\sum_{q\neq 0}\sum_{k=1}^{N}[v(k)]\int_{\beta(k)}^{\beta(k+1)}\mathfrak{D}^2(\lambda_q,s)ds$$
$$\oplus\sum_{k=1}^{M}[w(k)]\int_{\gamma(k)}^{\gamma(k+1)}ds\int_{1/4}^{\infty}\mathfrak{D}^2(\lambda^l,s)ds.$$

So it is sufficient to determine the multiplicities and the intervals of the integrals in each case.

V) $\mathfrak{R}=\mathfrak{D}^1(\rho(m),b_1)\otimes\mathfrak{D}^1(\rho(n),b_2)$, $(m\geq n)$.

i) When $b_1b_2>0$, $u=u(m,n,h)=(2n+1)(2h+1)$, if $0\leq h\leq m-n)$; $=(2m+1)(2n+1)-(m+n-h)(m+n-h+1)$, (if $m-n\leq h\leq m+n$); $=(2m+1)(2n+1)$, (if $m+n\leq h$).

$$(\alpha(1),\alpha(2))=(b_1+b_2,(\text{sign } b_1)\infty) \text{ or } ((\text{sign } b_1)\infty, b_1+b_2).$$

And $$v=w=0.$$

ii) When $b_1b_2<0$, $u=u(m,n,h)$ as in the case i), and $(\alpha(1),\alpha(2))=(0,b_1+b_2)$ or $(b_1+b_2,0)$.

$N=M=1$, and $v(1)=v(m,n,q)=(1/2)(m+n-|q|+1)(m+n-|q|+2)$, (if $m-n\leq|q|\leq m+n$); $=(2n+1)(m+|q|+1)$, (if $0<|q|\leq m-n$); $=0$, (otherwise).

$$w(1)=w(m,n)=(2m+1)(2n+1).$$
$$(\beta(1),\beta(2))=(\gamma(1),\gamma(2))=(0,\infty).$$

VI) $\mathfrak{R}=\mathfrak{D}^2(\lambda,c)\otimes\mathfrak{D}^1(\rho(m),b)$.

For $\lambda=\lambda^l$, $u=u(m,h)=(2m+1)(2h+1)$.

For $\lambda=\lambda_p$, $u=u(p,m,h)=(2m+1)(h-|p|+1)$, (if $|p|+m\leq h$); $=(1/2)(h-|p|+m+2)(h-|p|+m+1)$, (if $\max(m-|p|,0)\leq h\leq p+m$); $=(2h+1)(m-|p|+1)$, (if $0\leq h\leq m-|p|$).

And $u(I,m,h)=2m+1$, (if $m\leq h$); $=2h+1$, (if $0\leq h\leq m$).

$$(\alpha(1),\alpha(2))=(0,(\text{sign } b)\infty) \text{ or } ((\text{sign } b)\infty,0).$$

$N=M=1$, and for $\lambda=\lambda^l$, $v(1)=\infty$.

For $\lambda=\lambda_p$, $v(1)=v(m,q)=\infty$, (if $pq>0$); $=(1/2)(m-|p|-|q|+2)\times(m-|p|-|q|+1)$, (if $pq<0, 0<|q|\leq m-|p|$); $=0$, (otherwise).

$v(I,m,q)=m-|q|+1$, (if $0<|q|\leq m$); $=0$, (otherwise).

For $\lambda\neq I$, $w(1)=\infty$; and $w(I,m)=2m+1$.

$$(\beta(1),\beta(2))=(\gamma(1),\gamma(2))=(0,\infty).$$

VII) $\mathfrak{R}_+=\mathfrak{D}^3(\kappa,+)\otimes\mathfrak{D}^1(\rho(m),b)$. (resp. $\mathfrak{R}_-=\mathfrak{D}^3(\kappa,-)\otimes\mathfrak{D}(\rho(m),b)$).

i) $b>0$. (resp. $b<0$).

For $\kappa=\kappa^\rho$, $u=u(m,h)=(2m+1)(2h+1)$.

For $\kappa=\kappa_n$, $u=u(n,m,h)=2m+1$, (if $m+|n|\leq h$); $=m-|n|+h+1$, (if $\max(0,m-|n|)\leq h\leq m+|n|$); $=2h+1$, (if $0\leq h\leq m-|n|$).

$(\alpha(1),\alpha(2))=(b,\infty)$. (resp. $(-\infty,b)$). And $v=w=0$.

ii) $b<0$. (resp. $b>0$).

$u=u(\kappa, m, h)$, as in the case i).

$$(\alpha(1), \alpha(2))=(b, 0). \quad (\text{resp. } (0, b)).$$

$N=M=1$, and for $\kappa=\kappa^{\rho}$, $v(1)=\infty$.

For $\kappa=\kappa_n$, $v(1)=v(n, m, q)=m+n-q+1$, (if $\max(n-m, 1)\leq q\leq m+n$); $=m-n+q+1$, (if $n-m\leq q\leq \min(m+n, -1)$); $=0$, (otherwise). (resp. $=m+n+q+1$, (if $\max(n-m, 1)\leq -q\leq m+n$); $=m-n-q+1$, (if $n-m\leq -q\leq \min(m+n, -1)$); $=0$, (otherwise)).

For $\kappa=\kappa^{\rho}$, $w(1)=\infty$, and $w(\kappa_n, m)=2m+1$.

$$(\beta(1), \beta(2))=(\gamma(1), \gamma(2))=(0, \infty).$$

VIII) $\mathfrak{R}=\mathfrak{D}^2(\lambda, c_1)\otimes\mathfrak{D}^2(\mu, c_2)$.

For $(\lambda, \mu)=(\lambda^l, \lambda^l)$ or (λ^l, λ_p) or $(\lambda_p, \lambda_r\,(pr>0))$, $u=\infty$.

For (λ^l, I), $u=u(h)=2h+1$; and for $(\lambda_p, \lambda_r\,(pr<0))$, $u=u(p, r, h)=(1/2)\times(h-|r-p|+2)(h-|r-p|+1)$, (if $|r-p|\leq h$); $=0$, (otherwise).

For (λ_p, I), $u=u(p, h)=(h-|p|+1)$, (if $|p|\leq h$); $=0$, (otherwise).

And for (I, I), $u=1$.

$$(\alpha(1), \alpha(2))=(-\infty, \infty).$$

$N=3$, and for $(\lambda, \mu)=(\lambda^l, \lambda^l)$ or (λ^l, λ_p) or $(\lambda_p, \lambda_r\,(pr>0))$ or (λ^l, I), $v(1)=\infty$.

For $(\lambda, \mu)=(\lambda_p, \lambda_r\,(pr<0))$ or (λ_p, I), $v(1)=v(p, q)=\infty$, (if $(c_1-c_2)pq>0$); $=0$, (otherwise), for (I, I), $v(1)=0$.

For $(\lambda, \mu)\neq(I, I)$, $v(2)=\infty$; and for (I, I), $v(2)=0$.

For $(\lambda, \mu)=(\lambda^l, \lambda^l)$ or (λ^l, λ_p) or $(\lambda_p, \lambda_r\,(pr<0))$ or (λ^l, I), $v(3)=\infty$; and for $(\lambda_p, \lambda_r\,(pr>0))$ or (λ_p, I), $v(3)=v(p, q)=\infty$, (if $pq>0$); $=0$, (otherwise). For (I, I), $v(3)=0$.

$$(\beta(1), \beta(2), \beta(3), \beta(4))=(0, |c_1-c_2|, c_1+c_2, \infty).$$

$M=1$, and $w(1)=2$, (if $(\lambda, \mu)=(I, I)$); $=\infty$, (otherwise).

$$(\gamma(1), \gamma(2))=(0, \infty).$$

IX) i) $\mathfrak{R}_+=\mathfrak{D}^3(\kappa, +)\otimes\mathfrak{D}^2(\lambda, c)$.

For $(\kappa, \lambda)=(\kappa^{\rho}, \lambda^l)$ or $(\kappa^{\rho}, \lambda_p)$, $u=\infty$; and for (κ^{ρ}, I) or (κ_m, λ^l), $u=u(h)=2h+1$; for (κ_m, I), $u=u(h)=1$, (if $m\leq h$); $=0$, (otherwise).

For $(\kappa, \lambda)=(\kappa_m, \lambda_p)$, $u=u(m, p, h)=2h+1$, (if $0\leq h\leq |p-m|$, and $p(p-m)<0$); $=h+1+|p-m|$, (if $|p-m|\leq h$); $=0$, (otherwise).

$$(\alpha(1), \alpha(2))=(-\infty, 0).$$

$N=2$, and for $(\kappa, \lambda)=(\kappa^{\rho}, \lambda^l)$ or $(\kappa^{\rho}, \lambda_p)$ or (κ^{ρ}, I) or (κ_m, λ^l), $v(1)=\infty$.

For $(\kappa, \lambda)=(\kappa_m, \lambda_p)$, $v(1)=v(m, p, q)=\infty$, (if $pq>0$); $=|p-m+q|+1$, (if $|p-m|\geq|q|\geq 1, pq<0$, $p(p-m)<0$); $=0$, (otherwise).

For (κ_m, I), $v(1)=v(m, q)=1$, (if $1\leq|q|\leq|m|$, $mq<0$); $=0$, (otherwise).

For $(\kappa^{\rho}, \lambda^l)$ or $(\kappa^{\rho}, \lambda_p)$ or (κ^{ρ}, I) or (κ_m, λ^l), $v(2)=\infty$.

For (κ_m, λ_p), $v(2)=v(m, p, q)=\infty$, (if $pq>0$); $=|p+m-q|+1$, (if $|p+m|\geq|q|\geq 1, pq<0, p(p+m)<0$); $=0$, (otherwise).

For (κ_m, I), $v(2)=v(m, q)=1$, (if $1\leq|q|\leq|m|, mq>0$); $=0$, (otherwise).

$$(\beta(1), \beta(2), \beta(3))=(0, c, \infty).$$

$M=1$, and $w(1)=1$, (if $(\kappa, \lambda)=(\kappa_m, I)$); $=\infty$, (otherwise).

$$(\gamma(1), \gamma(2))=(0, \infty).$$

ii) $\mathfrak{R}_-=\mathfrak{D}^3(\kappa, -)\otimes\mathfrak{D}^2(\lambda, c)$.

For $(\kappa, \lambda)=(\kappa^\rho, \lambda^l)$ or (κ^ρ, λ_p), $u=\infty$, and for (κ^ρ, I) or (κ_m, λ^l), $u=u(h)=2h+1$.

For (κ_m, λ_p), $u=u(m, p, h)=h+|p+m|+1$, (if $|p+m|\leq h$); $=2h+1$, (if $0\leq h\leq|p+m|$, $p(p+m)<0$); $=0$, (otherwise).

For (κ_m, I), $u=u(m, h)=1$, (if $m\leq h$); $=0$, (otherwise).

$$(\alpha(1), \alpha(2))=(0, \infty).$$

$N=2$, and for $(\kappa, \lambda)=(\kappa^\rho, \lambda^l)$ or (κ^ρ, λ_p) or (κ^ρ, I) or (κ_m, λ^l), $v(1)=\infty$.

For (κ_m, λ_p), $v(1)=v(m, p, q)=\infty$, (if $pq>0$); $=|p+m-q|+1$, (if $|p+m|\geq|q|\geq 1, pq<0, p(p+m)<0$); $=0$, (otherwise).

For (κ_m, I), $v(1)=v(m, q)=1$, (if $1\leq|q|\leq|m|, mq>0$); $=0$, (otherwise).

For $(\kappa, \lambda)=(\kappa^\rho, \lambda^l)$ or (κ^ρ, λ_p) or (κ^ρ, I) or (κ_m, λ^l), $v(2)=\infty$, and for (κ_m, λ_p), $v(2)=v(m, p, q)=\infty$, (if $pq>0$); $=|p-m-q|+1$, (if $|p-m|\geq|q|\geq 1, pq<0, p(p-m)<0$); $=0$, (otherwise).

For (κ_m, I), $v(2)=v(m, q)=1$, (if $1\leq|q|\leq|m|, mq<0$); $=0$, (otherwise).

$$(\beta(1), \beta(2), \beta(3))=(0, c, \infty).$$

$M=1$, $w(1)=w(\kappa, \lambda)$, $(\gamma(1), \gamma(2))$, as in the case i).

X) $\mathfrak{D}^3(\kappa, +)\otimes\mathfrak{D}^3(\kappa, +)\cong[\infty]\int_0^\infty\mathfrak{D}^3(\kappa^\rho, +)d\rho$,

$$\mathfrak{D}^3(\kappa, +)\otimes\mathfrak{D}^3(\kappa, -)\cong[\infty]\int_0^\infty\mathfrak{D}^3(\kappa^\rho, +)d\rho\oplus[\infty]\int_0^\infty\mathfrak{D}^3(\kappa^\rho, -)d\rho,$$

$$\mathfrak{D}^3(\kappa, -)\otimes\mathfrak{D}^3(\kappa, -)\cong[\infty]\int_0^\infty\mathfrak{D}^3(\kappa^\rho, -)d\rho.$$

§ 3. Besides the above considered ordinary representations, there are so-called spinor representations of G. In the same way, we can treat the Kronecker products of two spinor representations or of a spinor representation and ordinary one. It is easy to obtain very analogous results, in which instead of integers half-integers appear and play important roles as the number denoting the kind of representations. We leave the details about these problems to another paper.

References

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