40. 2-Primary Components of the Homotopy Groups of Spheres

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This is a preliminary report of results concerning the generators of 2-primary components of the homotopy groups of spheres. The proofs will be given elsewhere.

1. Let E: $\pi_m(S^n)\to\pi_{m+1}(S^{n+1})$ be the suspension homomorphism. If α_n is an essential element of $\pi_m(S^n)$, we shall denote $E\alpha_n$ by α_{n+1}. The homotopy class of the identity map $S^n\to S^n$ is denoted by ι_n. Let $\alpha\in\pi_p(S^n), \beta\in\pi_q(S^n)$. Suppose that there exists a map $h: S^p\times S^q\to S^n$ of type (α,β), then we denote by (α,β) the coset of the subgroup $E\pi_{p+q}(S^n)$ of $\pi_{p+q+1}(S^{n+1})$ which includes the homotopy class of the map obtained by Hopf construction from h. The triad Whitehead product of α and β will be denoted by $\{\alpha,\beta\}\in\pi_{p+q+1}(S^{n+1}; E_+, E_-)$ ([4] or [5]). Define a homomorphism $P: \pi_m(S^n)\to\pi_{m+n+1}(S^{n+1}; E_+, E_-)$ by $P(\alpha)=\{\alpha,\iota_n\}$ for $\alpha\in\pi_m(S^n)(n\geqq 2)$.

We use the notation R_n instead of $SO(n)$. G. W. Whitehead defined a homomorphism $J: \pi_m(R_n)\to\pi_{m+n}(S^n)$ ([6]). We can prove

(1.1) $\qquad J(\alpha\circ\beta)=J(\alpha)\circ E^n\beta$ for $\beta\in\pi_p(S^m), \alpha\in\pi_m(R_n)$,

and that in the diagram:

$$
(1.2)\qquad
\begin{array}{ccccccc}
\cdots\pi_m(R_n) & \xrightarrow{j^*} & \pi_m(R_{n+1}) & \xrightarrow{p^*} & \pi_m(S_n) & \xrightarrow{\tilde{\Delta}} & \pi_{m-1}(R_n)\\
\downarrow J & & \downarrow J & & \downarrow P & & \downarrow J\\
\cdots\to\pi_{m+n}(S^n) & \xrightarrow{E} & \pi_{m+n+1}(S^{n+1}) & \xrightarrow{i^*} & \pi_{m+n+1}(S^{n+1}; E_+, E_-) & \xrightarrow{\Delta} & \pi_{m+n-1}(S_n)
\end{array}
$$

the upper sequence is the bundle sequence of $R_{n+1}\to R_{n+1}/R_n$, and the lower sequence is the suspension sequence of S^n. We can also prove that the following relations hold:

(1.3) $\qquad$ (a) $E\circ J=J\circ j^*$, (b) $i^*\circ J=P\circ p^*$ (c) $\Delta\circ P=-J\circ\tilde{\Delta}$.

I. M. James [4] defined a homomorphism $H: \pi_m(S^n)\to\pi_m(S^{2n-1})$, which is a generalization of the Hopf-invariant.

P. J. Hilton [7] also defined a homomorphism $\hat{H}: \pi_m(S^n)\to\pi_m(S^{2n-1})$ in a different way. I owe M. G. Barratt the announcement that H is the same as $\hat{H}$.

We denote by $\{\alpha,\beta,\gamma\}$ and $\{\alpha,\beta,\gamma,\delta\}$ Toda's constructions, ([1], [2], [3]).

The homotopy groups $\pi_{n+r}(S^n), r\leqq 7$, are well known. We list the generators of 2-primary components of them as follows:

$$\iota_n \in \pi_n(S^n)(n\geqq 1),\ \eta_n \in \pi_{n+1}(S^n)(n\geqq 2),\ \alpha_n \in \pi_{n+3}(S^n)(n\geqq 3),$$
$$(1.4)\qquad \nu_n \in \pi_{n+3}(S^n)(n\geqq 4),\ [\iota_6, \iota_6] \in \pi_{11}(S^6),\ \beta_5'' \in \pi_{12}(S^5),$$
$$\beta_6' \in \pi_{13}(S^6),\ \beta_n \in \pi_{n+7}(S^n)(n=7,8),\ \mu_n \in \pi_{n+7}(S^n)(n\geqq 8),$$

where $\eta_2 \in (\iota_1, \iota_1)$, $\alpha_3 \in (\eta_2, \iota_2)$, $\nu_4 \in (\iota_3, \iota_3)$, $\beta_5'' \in (12\nu_4, \iota_4)$
$\beta_6 \in (\eta_5 \circ \eta_6, \iota_5)$, $\beta_7 \in (\eta_6, \iota_6)$, and $\mu_8 \in (\iota_7, \iota_7)$.

The other generators of 2-primary components of $\pi_{n+r}(S^n)(r\leqq 7)$ can be represented as the compositions of the above elements.

The relations between these elements are also well known. We only add the following:

$$(1.5)\qquad \begin{array}{lll} \text{(a)} & \{\eta_n, 8\iota_{n+1}, \nu_{n+1}\} \equiv 0 & \bmod 0\ (n\geqq 5). \\ \text{(b)} & \{\nu_5, 8\iota_8, \nu_8\} \equiv \beta_5'' & \bmod 0. \\ \text{(c)} & \{\eta_n, \nu_{n+1}, \eta_{n+4}\} \equiv \nu_n \circ \nu_{n+3} & \bmod 0\ (n\geqq 5). \end{array}$$

2. In the following sections, we always consider 2-primary components of groups. For simplicity, we shall denote e.g. $\pi_m(S^n)$ to mean the 2-primary components of $\pi_m(S^n)$; and use the terms such as "equal", "isomorphic", in the sense of C_2. ([10]).

$\pi_{n+r}(S^n)$, $r\leqq 13$ were calculated by H. Toda [1]. We can obtain generators of these groups and the relations between them by using the diagram (1.2), Hopf homomorphism, expansion of Whitehead products [8], and Toda's formula [9].

In the following tables, ∞ means an infinite cyclic group, 2^n means a cyclic group of order 2^n. These tables are read as follows: e.g. in the columns $n=5, 6$, and in the entry $r=8$, we have 2, and 2, 8. This means $\pi_{8+5}(S^5)=z_2$, $\pi_{8+6}(S^6)=z_2+z_8$, and the first z_2 in $\pi_{14}(S^6)$ is induced by the suspension homomorphism from z_2 in $\pi_{13}(S^5)$.

Table I. $\pi_{n+r}(S^n)$, $8\leqq r\leqq 10$

$n=$	2	3	4	5	6	7	8	9	10	11	12	Generators
$r=8$	0	2	2	2	2	2	2	2	2	2	2	ε_n
					8	2	2	2				ν'_n
						2	2	0	0	0	0	$\beta_n \circ \eta_{n+7}$
							2	2	↘2	2	2	$\mu_n \circ \eta_{n+7}$
$r=9$	2	2	2	2	2	2	2	2	↗2	2	2	$\eta_n \circ \varepsilon_{n+1}$
		2	2	2	2	2	2	2	2	2	2	δ_n
			2	2	2	2	2	2				$\nu_n \circ \nu_{n+3} \circ \nu_{n+6}$
							2	2	↘2	2	2	$\mu_n \circ \eta_{n+7} \circ \eta_{n+8}$
						2	2	0	0	0	0	$\beta_n \circ \eta_{n+7} \circ \eta_{n+8}$
								∞	0	0	0	$[\iota_{10}, \iota_{10}]$
$r=10$	2	2	2	2	2	2	2	2	2	2	2	$\eta_n \circ \delta_{n+1}$
	2											$\eta_2 \circ \eta_3 \circ \varepsilon_4$
		4	4									ε'_n
			8									$\nu_4 \circ \beta_7$
				8	8	8	8					$\nu_n \circ \mu_{n+3}$
							8	↘8	4	2	0	$\mu_n \circ \nu_{n+7}$

In the Table I, we omit $\pi_{1+r}(S^1)$, because they are always 0 for $r\geqq 1$.
Relations:

(2.1) $H(\varepsilon_3)=\nu_5\circ\nu_8,\ H(\nu'_6)=\nu_{11},\ H(\delta_3)=\beta''_5,\ H(\varepsilon'_3)=\varepsilon_5,\ H[\iota_{10},\iota_{10}]=-2\iota_{19}.$

(2.2) $[\iota_9,\iota_9]=\nu'_9+\mu_9\circ\eta_{16},\ [\iota_{11},\iota_{11}]=\mu_{11}\circ\nu_{18},\ [\iota_6,\iota_6]\circ\nu_{11}=[\nu_6,\iota_6]=2\nu'_6,$
$[\iota_{10},\iota_{10}]\circ\eta_{19}=[\eta_{10},\iota_{10}]=2\mu_{10}\circ\nu_{19}.$

(2.3) $\eta_4\circ\beta''_5=0,\ \beta''_5\circ\eta_{12}=0,\ \eta_5\circ\beta'_6=0,\ \eta_6\circ\beta_7=\beta'_6\circ\eta_{13}=4\nu'_6,$
$\eta_n\circ\mu_{n+1}=\mu_n\circ\eta_{n+7}+\nu'_n(n=7,8),\ \eta_9\circ\mu_{10}=\nu'_9,$
$\eta_n\circ\mu_{n+1}=\mu_n\circ\eta_{n+7}=\nu'_n(n\geqq 10),$
$\varepsilon_n\circ\eta_{n+8}=\eta_n\circ\varepsilon_{n+1}\ (n\geqq 3),$
$\nu'_n\circ\eta_{n+8}=\eta_n\circ\nu'_{n+1}\ (n\geqq 6),$
$\eta_n\circ\nu'_{n+1}=\nu_n\circ\nu_{n+3}\circ\nu_{n+6}+\eta_n\circ\varepsilon_{n+1}\ (n\geqq 5),$
$\eta_n\circ\delta_{n+1}=\delta_n\circ\eta_{n+9}\ (n\geqq 3),$
$2\varepsilon'_3=\eta_3\circ\eta_4\circ\varepsilon_5,\ \alpha_3\circ\beta'_6=0,\ \alpha_4\circ\beta_7=2\varepsilon'_4,$
$\varepsilon'_5=2(\nu_5\circ\mu_8)=\nu_5\circ\beta_8,\ \beta''_5\circ\nu_{12}=4(\nu_5\circ\mu_8),\ \beta'_6\circ\nu_{13}=-2(\nu_6\circ\mu_9),$
$\beta_7\circ\nu_{14}=-\nu_7\circ\mu_{10},\ \eta_9\circ[\iota_{10},\iota_{10}]=4(\mu_9\circ\nu_{16}),\ 2(\mu_9\circ\nu_{16})=\nu_9\circ\mu_{12}.$

Note that $\eta_n\circ\eta_{n+1}\circ\varepsilon_{n+2}=0\ (n\geqq 9),\ \varepsilon'_n=0\ (n\geqq 10),$
$\nu_n\circ\mu_{n+3}=0\ (n\geqq 11),\ \mu_n\circ\nu_{n+7}=0\ (n\geqq 12).$

(2.4) $\varepsilon_3\in\{\alpha_3,\nu_6,\nu_9\}=\{\eta_3,\alpha_4,\nu_7\}$ mod 0,
$\varepsilon_n\in\{\eta_n,2\iota_{n+1},\nu_{n+1}\circ\nu_{n+4}\}$ mod F $(n\geqq 4)$,
where $F=0$ if $n=4,5$, $F=(4\nu'_6)$ if $n=6$,
$F=(\eta_n\circ\mu_{n+1})$ if $n\geqq 7$.

$\nu'_6\in(\nu_5,\iota_5)$ mod (ε_6),
$\nu'_n+\varepsilon_n\in\{\nu_n,\eta_{n+3},\nu_{n+4}\}$ mod F $(n\geqq 6)$,
where $F=(2\nu'_6)$ if $n=6$, $F=0$ if $n\geqq 7$.

$\delta_4\in\{\eta_4,2\iota_5,\beta''_5\}$ mod $(\eta_4\circ\varepsilon_5)$,
$\delta_n\in\{\eta_n,2\iota_{n+1},\beta''_{n+1}\}$ mod $F\,(n\geqq 5)$,
where $F=(\eta_n\circ\varepsilon_{n+1},\nu_n\circ\nu_{n+3}\circ\nu_{n+6})$ if $n=5,6,9$,
$F=(\eta_n\circ\varepsilon_{n+1},\nu_n\circ\nu_{n+3}\circ\nu_{n+6},\beta_n\circ\eta_{n+7}\circ\eta_{n+8})$ if $n=7,8$.

$\varepsilon'_3\in\{\alpha_3,2\nu_6,\nu_9\}$ mod 0,
$\varepsilon'_3\in\{\eta_3,2\iota_4,\varepsilon_4\}=\{\varepsilon_3,2\iota_{11},\eta_{11}\}$ mod $(\eta_3\circ\eta_4\circ\varepsilon_5,\eta_3\circ\delta_4)$,
$\varepsilon'_n\in\{\eta_n,2\iota_{n+1},\varepsilon_{n+1}\}$ mod $F\,(n\geqq 4)$,
where $F=(\eta_4\circ\eta_5\circ\varepsilon_6,\eta_4\circ\delta_5)$ if $n=4$,
$F=(4\nu_n\circ\mu_{n+3})$ if $n=5,6,7,8$,
$F=(4\mu_9\circ\nu_{16})$ if $n=9$, $F=0$ if $n\geqq 10$.

$\{\nu_5,2\nu_8,\nu_{11}\}=\pi_{15}(S^5)$.
$\delta_3\in\{\eta_3,\alpha_4,8\iota_7,\nu_7\}$ mod $(\eta_3\circ\varepsilon_4)$.

Table II. $\pi_{n+r}(S^n)$, $11 \leqq r \leqq 13$

$n=$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Generators
$r=11$	4	0	0	0	0	0	0	0	0	0	0	0	0	0	$\eta_2 \circ \varepsilon'_3$
	2 ↘														$\eta_2 \circ \eta_3 \circ \delta_4$
		4	4 ↘												δ'_n
			8	8	8	8	8	8	8	8	8	8	8	8	ζ_n
		2	2	0	0	0	0	0	0	0	0	0	0	0	$\alpha_n \circ \varepsilon_{n+3}$
		2	2	0	0	0	0	0	0	0	0	0	0	0	$\varepsilon_n \circ \nu_{n+8}$
			2	0	0	0	0	0	0	0	0	0	0	0	$\nu_4 \circ \beta_7 \circ \eta_{14}$
			2	2	0	0	0	0	0	0	0	0	0	0	$\nu_n \circ \nu'_{n+3}$
			2	2 ↘											$\nu'_n \circ \varepsilon_{n+3}$
					4	2	2	2	0	0	0	0	0	0	$\nu'_n \circ \nu_{n+8}$
											∞	0	0	0	$[\iota_{12}, \iota_{12}]$
$r=12$	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$\eta_2 \circ \alpha_3 \circ \varepsilon_6$
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$\eta_2 \circ \varepsilon_3 \circ \nu_{11}$
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	$\eta_2 \circ \delta'_3$
		2	2	0	0	0	0	0	0	0	0	0	0	0	$\alpha_n \circ \delta_{n+3}$
		2	2	0	0	0	0	0	0	0	0	0	0	0	$\alpha_n \circ \eta_{n+3} \circ \varepsilon_{n+4}$
			2	0	0	0	0	0	0	0	0	0	0	0	$\nu_4 \circ \beta_7 \circ \eta_{14} \circ \eta_{15}$
			2	2	0	0	0	0	0	0	0	0	0	0	$\nu_n \circ \eta_{n+3} \circ \varepsilon_{n+4}$
			2	2	0	0	0	0	0	0	0	0	0	0	$\nu_n \circ \nu_{n+3} \circ \nu_{n+6} \circ \nu_{n+9}$
			2	2 ↘											$\nu_n \circ \delta_{n+3}$
					16	0	0	0	0	0	0	0	0	0	$[\iota_6, \iota_6] \circ \mu_{11}$
									4	0	0	0	0	0	$[\iota_{10}, \iota_{10}] \circ \nu_{19}$
										2	2	0	0	0	λ_n
											2	2	0	0	τ_n
$r=13$	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$\eta_2 \circ \alpha_3 \circ \delta_6$
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$\eta_2 \circ \alpha_3 \circ \eta_6 \circ \varepsilon_7$
		2	2	0	0	0	0	0	0	0	0	0	0	0	$\alpha_n \circ \eta_{n+3} \circ \delta_{n+4}$
			2	2	0	0	0	0	0	0	0	0	0	0	$\nu_n \circ \eta_{n+3} \circ \delta_{n+4}$
			8	0	0	0	0	0	0	0	0	0	0	0	$\nu_4 \circ \nu_7 \circ \mu_{10}$
				2	2	2	2	0	0	0	0	0	0	0	$\nu_n \circ \mu_{n+3} \circ \nu_{n+10}$
							2	2	2	2	0	0	0	0	$\mu_n \circ \mu_{n+7} \circ \nu_{n+10}$
										2	2	0	0	0	$\lambda_n \circ \eta_{n+12}$
											2	2	0	0	$\tau_n \circ \eta_{n+12}$
													∞	0	$[\iota_{14}, \iota_{14}]$

Relations:

(2.5) $H(\delta'_3)=\delta_5$, $H(\zeta_5)=8\mu_9$, $H(\lambda_{11})=\eta_{21}\circ\eta_{22}$, $H(\tau_{12})=\eta_{23}$,
$H[\iota_{12}, \iota_{12}]=-2\iota_{23}$, $H[\iota_{14}, \iota_{14}]=-2\iota_{27}$.

(2.6) $2\delta'_n=\eta_n\circ\eta_{n+1}\circ\delta_{n+2} (n\geqq 3)$, $2\zeta_n=\delta'_n (n\geqq 5)$,
$2\nu'_6\circ\nu_{14}=\nu_6\varepsilon_9$, $\nu_7\circ\varepsilon_{10}=0$, $\varepsilon'_n\circ\eta_{n+10}=\alpha_n\circ\varepsilon_{n+3}$ $(n=3, 4)$,
$\alpha_n\circ\nu'_{n+3}=\varepsilon_n\circ\nu_{n+8}+\alpha_n\circ\varepsilon_{n+3}$ $(n=3, 4)$,
$\nu_n\circ\mu_{n+3}\circ\eta_{n+10}=\nu_n\circ\varepsilon_{n+3}$ $(n=5, 6)$,
$\eta_4\circ\zeta_5=\alpha_4\circ\delta_7$, or $\alpha_4\circ\delta_7+\alpha_4\circ\eta_7\circ\varepsilon_8$,
$[\iota_6, \iota_6]\circ\mu_{11}\neq 0$, $[\iota_{10}, \iota_{10}]\circ\nu_{19}\neq 0$,
$\nu_7\circ[\iota_{10}, \iota_{10}]=[\nu_7, \nu_7]=0$, $\eta_{11}\circ[\iota_{12}, \iota_{12}]=[\eta_{11}, \eta_{11}]=0$, $[\iota_{12}, \iota_{12}]\circ\eta_{23}=\lambda_{12}$,
$\delta_n\circ\nu_{n+9}=\alpha_n\circ\eta_{n+3}\circ\varepsilon_{n+4}$ $(n=3, 4)$,
$\delta'_n\circ\eta_{n+11}=\alpha_n\circ\delta_{n+3}$ $(n=3, 4)$,
$\zeta_n\circ\eta_{n+11}=\nu_n\circ\delta_{n+3}$ $(n=5, 6)$,
$\eta_{10}\circ\lambda_{11}=0$, $\eta_{11}\circ\tau_{12}=\mu_{11}\circ\nu_{18}\circ\nu_{21}+\lambda_{11}\circ\eta_{23}$,
$[\iota_6, \iota_6]\circ\varepsilon_{11}=[\varepsilon_6, \iota_6]=0$, $[\iota_6, \iota_6]\circ\nu'_{11}=[\nu'_6, \iota_6]=0$.

(2.7) $\delta'_n\in\{\eta_n, 2\iota_{n+1}, \delta_{n+1}\}$ mod F $(n\geqq 3)$,
where $F=(\eta_3\circ\eta_4\circ\delta_5, \eta_3\circ\varepsilon_4)$ if $n=3$,

$F=(\eta_n \circ \eta_{n+1} \circ \delta_{n+2})$ if $n \geqq 4$.

$\delta'_n \in \{\delta_n, 2\iota_{n+9}, \eta_{n+9}\}$ mod F' $(n \geqq 3)$,

where $F'=(\eta_n \circ \eta_{n+1} \circ \delta_{n+2}, \alpha_n \circ \varepsilon_{n+3})$ if $n=3,4$,

$F'=(\eta_n \circ \eta_{n+1} \circ \delta_{n+2}, \nu_n \circ \varepsilon_{n+3})$ if $n=5,6$,

$F'=F$ $(n \geqq 7)$.

$\zeta_n \in \{\nu_n, 8\iota_{n+3}, \beta_{n+3}\}$ mod F $(n \geqq 5)$,

where $F=(\nu_5 \circ \nu'_8, \nu_5 \circ \varepsilon_8)$ if $n=5$,

$F=(\nu_6 \circ \nu_9)$, if $n=6$, $F=0$ if $n \geqq 7$.

$\lambda_n \in \{\mu_n, 2\nu_{n+7}, \eta_{n+10}\}$ mod 0 $(n=11, 12)$.

$\tau_n \in \{\mu_n, \nu_{n+7}, \eta_{n+10}\}$ mod F $(n=12, 13)$,

where $F=(\lambda_{12})$ if $n=12$, $F=0$ if $n=13$.

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