98. The Product of a Logarithmic Method and the Sequence-to-Sequence Quasi-Hausdorff Method

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1. *Definitions.* μ_n, defined by

(1.1) $$\mu_n=\int_0^1 t^n d\chi(t) \quad (n=0, 1, 2, \cdots),^*$$

where $\chi(t)$ is a real function of bounded variation in (0, 1), is called the *moment constant* of *rank* n generated by the *mass-function* $\chi(t)$. If, further,

(1.2) $$\chi(1)=1, \ \chi(+0)=\chi(0)=0,^{+}$$

μ_n is said to be a *regular* moment constant.
The matrix $\lambda\equiv(H, \mu_n)$, defined by

(1.3) $$\lambda_{nk}=\begin{cases}\binom{n}{k}\Delta^{n-k}\mu_k & (n\geq k)\\ 0 & (n<k),\end{cases}$$

is termed the *Hausdorff-matrix* corresponding to the sequence of moment constants $\{\mu_n\}$. The summability (H, μ_n) of a sequence $\{s_n\}$ to the sum s is defined as the convergence to a finite limit s of its *Hausdorff transform*, or simply (H, μ_n) transform, σ_n, where

(1.4) $$\sigma_n=\sum_{k=0}^{n}\lambda_{nk}\, s_k \quad (n=0, 1, 2, \cdots).$$

The transpose of the Hausdorff matrix, that is, the matrix $\lambda^*\equiv(H^*, \mu_n)$, defined by

(1.5) $$\lambda^*_{nk}=\begin{cases}\binom{k}{n}\Delta^{k-n}\mu_n & (n\leq k)\\ 0 & (n>k)\end{cases}$$

is termed the *Quasi-Hausdorff matrix* corresponding to the sequence of moment constants $\{\mu_n\}$.

The sequence-to-sequence *Quasi-Hausdorff transform*, or simply the (H^*, μ_n) transform, $\sigma_n{}^*$ of a sequence $\{s_n\}$ is defined by

(1.6) $$\sigma_n{}^*=\sum_{k=n}^{\infty}\binom{k}{n}\Delta^{k-n}\mu_n\, s_k.$$

Since μ_n is given by (1.1), we also have

(1.7) $$\sigma_n{}^*=\sum_{k=n}^{\infty}\int_0^1 s_k\binom{k}{n}t^n(1-t)^{k-n}d\chi(t).$$

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* The function t^0 is defined at $t=0$ so as to be continuous; thus

$$\mu_0=\int_0^1 d\chi(t).$$

+ The assumption $\chi(0)=0$ is not a substantial restriction.

The summability (H^*, μ_n) of a sequence $\{s_n\}$ to the limit s is defined as the convergence to a finite limit s of its (H^*, μ_n) transform $\sigma_n{}^*$.

It is well-known (see [2], Theorem 217, p. 217, p. 276) that the necessary and sufficient conditions that the (H, μ_n) transform be *regular*, that is, $\sigma_n \to s$ whenever $s_n \to s$, are:

$$\chi(1)=1, \ \chi(+0)=\chi(0)=0. \tag{1.8}$$

We also know that if μ_n is a regular moment constant generated by the mass-function $\chi(t)$, then the necessary and sufficient conditions that the (H^*, μ_n) transform be regular are (see [2], Theorem 219, p. 279; also notes on chapter XI, § 11. 20)

$$\begin{cases} \text{(i)} \quad \int_0^1 \frac{|d\chi(t)|}{t} \leq k < \infty, ^{4)} \\ \text{(ii)} \quad \int_0^1 \frac{d\chi(t)}{t} = 1. \end{cases} \tag{1.9}$$

Borwein [1] has recently defined the following logarithmic method of summability, denoted by L.

A sequence $\{s_n\}$ is said to be *summable* (L) to s, if

$$L_s(x) = -\{\log(1-x)\}^{-1} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} \tag{1.10}$$

tends to a finite limit s as $x \to 1-0$ in the right open interval $(0, 1)$.

2. *Introduction.* Let A and B be two summability methods for sequences $\{s_n\}$, and let us denote by AB the iteration-product which associates with any given sequence the A-transform of its B-transform (of course, provided it is possible to define it).

The question of determining under what circumstances $A \subseteq AB$, that is, A-summability implies AB-summability, was raised by Szász [9] in 1952 at the suggestion of Prof. I. M. Sheffer. And Szász himself demostrated in a couple of papers ([9], [10]) the truth of a number of inclusion relations of this type by considering various pairs of summability methods. Subsequently, this line of study has been taken up by several workers like Rajagopal [8], Pati [5], Ramanujan [6], Jakimovski [3], Borwein [1] and Lal [4].

Borwein has established the inclusion relation:

$$(L) \subseteq (L)(H, \mu_n)$$

in the case in which (H, μ_n) is regular.

The object of the present paper is to establish an inclusion relation between summability (L) and the product of summability (L) and the Quasi-Hausdorff method of summability (H^*, μ_n), which corresponds to a mass-function, satisfying certain general conditions.

3. We shall write throughout

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4) K always denotes an absolute constant, not necessarily the same at each ocurence.

$$\chi^*(t)=\int_0^t \frac{|d\chi(u)|}{u}.$$

We establish the following theorem.

Theorem. If (H^*, μ_n) be a regular sequence-to-sequence Quasi-Hausdorff method and the mass-function $\chi(t)$ generating μ_n satisfies the conditions:

(3.1) $$\chi^*(\eta)=0(1), \text{ as } \eta\to 0,$$

and

(3.2) $$\int_0^\eta \frac{\log\left(\frac{1}{t}\right)}{t}|d\chi(t)|=O\left(\log\frac{1}{\eta}\right), \quad \text{as } \eta\to 0.$$

then $$(L)\subseteq(L)(H^*, \mu_n).$$

Proof of the theorem.

We assume that $L_s(x)\to s$, as $x\to 1-0$. We then have to prove that, under the hypotheses of the theorem,

$$L_{\sigma^*}(x)\to s, \quad \text{as} \quad x\to 1-0.$$

We have, by (1.7),

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\sigma_n{}^*}{n+1}x^{n+1}\\
&=\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}\sum_{k=n}^{\infty}\int_0^1 s_k\binom{k}{n}(1-t)^{k-n}t^{n+1}\frac{d\chi(t)}{t}\\
&=\int_0^1\left\{\sum_{k=0}^{\infty}\frac{s_k}{k+1}x^{k+1}\sum_{n=0}^{k}\binom{k+1}{n+1}(1-t)^{k-n}t^{n+1}x^{n-k}\right\}\frac{d\chi(t)}{t}\\
&=\int_0^1\left\{\sum_{k=0}^{\infty}\frac{s_k}{k+1}x^{k+1}\left[\left(\frac{1-t}{x}+t\right)^{k+1}-\left(\frac{1-t}{x}\right)^{k+1}\right]\right\}\frac{d\chi(t)}{t}\\
&=\int_0^1\left\{\sum_{k=0}^{\infty}\frac{s_k}{k+1}[(1-ty)^{k+1}-(1-t)^{k+1}]\right\}\frac{d\chi(t)}{t},
\end{aligned}$$

on writing $1-x=y$.

Set now

$$g(t)=-\sum_{n=0}^{\infty}\frac{s_n}{n+1}(1-t)^{n+1};\ f(t)=g(t)/\log t.$$

We have,

$$\sum_{n=0}^{\infty}\frac{\sigma_n{}^*}{n+1}x^{n+1}=\int_0^1 g(t)\frac{d\chi(t)}{t}-\int_0^1 g(yt)\frac{p\chi(t)}{t}$$

and therefore,

$$\begin{aligned}
&-\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{\sigma_n{}^*}{n+1}x^{n+1}\\
&=\frac{1}{\log y}\int_0^1 g(yt)\frac{d\chi(t)}{t}-\frac{1}{\log y}\int_0^1\frac{g(t)}{t}d\chi(t)\\
&=\frac{1}{\log y}\int_0^1 f(yt)\log yt\frac{d\chi(t)}{t}-\frac{1}{\log y}\int_0^1 f(t)\log t\frac{d\chi(t)}{t}
\end{aligned}$$

$$=\frac{1}{\log y}\int_0^1[f(yt)-s]\log(yt)\frac{d\chi(t)}{t}-\frac{1}{\log y}\int_0^1[f(t)-s]\log t\,\frac{d\chi(t)}{t}+s$$
$$=I_1(y)-I_2(y)+s,$$

using the condition (1.9) (ii), which holds in view of the regularity of (H^*, μ_n). In view of the last step there occurs no loss of generality in assuming that $s=0$.

All the inversions involved in the foregoing steps are justified by virtue of the absolute convergence of the integrals $I_1(y)$ and $I_2(y)$, the proof of which fact is contained in that of (3.3).

We show below that, if $f(t)=0(1)$, as $t\to 0$, then, with $s=0$,

(3.3) $$\left.\begin{matrix}I_1(y)\\ I_2(y)\end{matrix}\right\}=0(1),\quad \text{as}\quad y\to 0.$$

We have, with $s=0$,

$$I_1(y)\equiv\int_0^1 f(yt)\frac{\log\frac{1}{yt}}{\log\frac{1}{y}}\frac{d\chi(t)}{t}$$
$$=\int_0^y f(yt)\frac{\log\frac{1}{yt}}{\log\frac{1}{y}}\frac{d\chi(t)}{t}=\int_y^1 f(yt)\frac{\log\frac{1}{yt}}{\log\frac{1}{y}}\frac{d\chi(t)}{t}$$
$$=I_{11}(y)+I_{12}(y),\quad \text{say}.$$

We now introduce an *additional notation*.

$$m(\tau)\equiv\max_{0<t<\tau}|f(t)|,$$

so that $m(\tau)\to 0$, as $\tau\to 0$. Also $m(\tau)$ is a non-decreasing function of τ.

We have

$$|I_{11}(y)|\le\int_0^y|f(yt)|\frac{|d\chi(t)|}{t}+\int_0^y|f(yt)|\frac{\log\frac{1}{t}}{\log\frac{1}{y}}\frac{|d\chi(t)|}{t}$$
$$\le m(y)\int_0^y\frac{|d\chi(t)|}{t}+m(y)\int_0^y\frac{\log\frac{1}{t}}{\log\frac{1}{y}}\frac{|d\chi(t)|}{t}$$
$$=m(y)[I_{111}(y)+I_{112}(y)].$$

Evidently

$$m(y)I_{111}(y)=0(1),\quad \text{as } y\to 0.$$

Also, by hypothesis,

$$\frac{1}{\log\frac{1}{y}}\int_0^y\log\frac{1}{t}\frac{|d\chi(t)|}{t}=O(1),\quad \text{as } y\to 0,$$

and, therefore,

$$m(y)I_{112}(y)=0(1),$$

as $y\to 0$.

Hence

$$I_{11}(y)=0(1), \quad \text{as } y\to 0.$$

Again, we have

$$\begin{aligned}
|I_{12}(y)| &\equiv \left| \int_y^1 f(yt) \frac{\log\frac{1}{yt}}{\log\frac{1}{y}} \frac{d\chi(t)}{t} \right| \\
&\leq 2\int_y^1 |f(yt)| \frac{|d\chi(t)|}{t} \quad [\text{since } y^2<yt\leq y<1] \\
&\leq 2\left[\int_y^{1/(y+1)} |f(yt)| \frac{|d\chi(t)|}{t} + \int_{1/(y+1)}^1 \frac{|d\chi(t)|}{t}\right] \\
&\leq 2\left[m\left(\frac{y}{1+y}\right)\int_y^{1/y+1}\frac{|d\chi(t)|}{t} + m(y)\int_{1/1+y}^1 \frac{|d\chi(t)|}{t}\right] \\
&=0(1),
\end{aligned}$$

as $y\to 0$.

We now have only to show that, with $s=0$,

$$I_2(y)=0(1), \quad \text{as } y\to 0.$$

$$\begin{aligned}
I_2(y) &\equiv \frac{1}{\log\frac{1}{y}}\int_0^1 f(t)\log\frac{1}{t}\frac{d\chi(t)}{t} \\
&= \frac{1}{\log\frac{1}{y}}\int_0^y f(t)\log\frac{1}{t}\frac{d\chi(t)}{t} + \frac{1}{\log\frac{1}{y}}\int_y^1 f(t)\log\frac{1}{t}\frac{d\chi(t)}{t} \\
&= I_{21}(y)+I_{22}(y), \text{ say.}
\end{aligned}$$

Now

$$\begin{aligned}
|I_{21}^{(y)}| &\leq \frac{1}{\log\frac{1}{y}}\int_0^y |f(t)|\log\frac{1}{t}\frac{|d\chi(t)|}{t} \\
&\leq m(y)\frac{1}{\log\frac{1}{y}}\int_0^y \log\frac{1}{t}\frac{|d\chi(t)|}{t} \\
&=0(1),
\end{aligned}$$

as $y\to 0$, since we have assumed that

$$\frac{1}{\log\frac{1}{y}}\int_0^y \log\frac{1}{t}\frac{|d\chi(t)|}{t}=O(1),$$

as $y\to 0$.

Now, integrating by parts,

$$|I_{22}(y)| \leq k\frac{1}{\log\frac{1}{y}}\int_y^1 \log\frac{1}{t}\frac{|d\chi(t)|}{t}$$

$$\leq k\chi^*(y)+0\left(\frac{1}{\log\frac{1}{y}}\int_y^1\frac{dt}{t}\right)$$
$$=0(1)+0(1)=0(1), \text{ as } y\to 0,$$

by hypothesis (3.1).

We have thus demonstrated the truth of (3.3), so that the theorem is established.

4. *Remarks.* It may be observed that the conditions (3.1) and (3.2) on the mass-function $\chi(t)$ are obviously satisfied for the following well-known special cases:

(i) The 'Circle' method (γ, a) (See Hardy [2], § 9.11 and § 11.21, [1]) for which

$$\chi(t)=\begin{cases} 0 & \text{for} \quad 0\leq t<a<1; \\ a & \text{for} \quad a\leq t<1. \end{cases}$$

(ii) The methods defined by

$$\chi(t)=lt^{l+1}/(l+1) \quad (l>0),$$

which are all equivalent to each other and to $(C, 1)$ for different positive values of l. (See Hardy, [2], § 11.21, [3].)

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