11. On the Radiation Pressure Exerted on a Non-Stationary Gaseous Cloud due to a Resonance Line

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1. *Introduction.* The radiation pressure exerted to a gaseous cloud by a neighbouring exciting star of high temperature has been discussed by Ambartsumian (1932, 1933), Zanstra (1934, 1949, 1950, 1951), Gerasimovič (1934), Chandrasekhar (1935, 1945a, 1945b), McCrea and Mitra (1936), Hagihara (1938, 1943), Hagihara and Hatanaka (1941, 1942), and others in view of application to problems of planetary nebulae, Wolf-Rayet stars and Be stars. They all assumed that the atoms and ions taking part in the radiative processes are at rest. In the present note the effect of the systematic and the random motions of the atoms is taken into consideration in discussing the radiation pressure. In such problems the central question is in the form of the line profiles. Also the question on the non-coherency of the absorbed and the emitted radiations comes in. The radiation incident to the cloud is assumed to be of Doppler's line profile and the probabilities for the absorption and the emission of radiation are taken to be of radiation damping form. It is assumed also the complete non-coherency of the radiation. The question for the radiative transfer is treated in view of application to thick gaseous clouds by the author's former method of using Green's function (1943). The atoms are supposed to have both systematic and random thermal motions. Even in the case of a resonance line the integration becomes complicated and I have to expand according to the powers of the thermal velocity of the atoms. In such a case of the resonance line treated in the present note, in which the distributions of the temperature and the density are supposed to be uniform throughout the cloud, the absorption of the direct radiation from the exciting star plays the main part of the radiation pressure. It is in this limited sense that the present note is dealing with due to the difficulty of the integration.

2. *Line Absorption.* Suppose that the atoms of the cloud are moving with velocity $V+v\cos\Theta$ in the positive z-direction, where V is of the systematic motion and v is of the random motion making an angle Θ with the positive z-direction. Suppose further that the distribution of the random velocities is such that the number of atoms in the ground state per unit volume is expressed by

$$N_1(v)=2\sqrt{\frac{M}{2\pi kT}}e^{-Mv^2/2kT}dv\frac{d\Omega}{4\pi},$$

where we have for isotropy $d\Omega/4\pi=\frac{1}{2}\sin\Theta\ d\Theta$, and M is the mass of an atom, k the Boltzmann constant and T is the temperature. Suppose that a radiation of intensity $I(\nu)d\nu$ is incident to the atoms in the positive z-direction and that an atom is excited from the ground state 1 to the excited state 2 by absorbing a light quantum $h\nu'$ as seen from the moving atom. The absorbed energy per unit volume per unit time is, as seen from the moving atom,

$$h\nu' B_{12}(\nu')N_1(v)I(\nu')d\nu',$$

where $B_{12}(\nu')$ is the transition probability for absorption.
Suppose that

$$B_{12}(\nu')=B_{12}\cdot\frac{a}{\pi}\frac{1}{a^2+(\nu'-\nu_0')^2}$$

and

$$I(\nu)=I\cdot\frac{1}{\sqrt{\pi}(\Delta\nu_D)}e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2},$$

where ν_0 is the frequency of the line center, $\Delta\nu_D$ is the Doppler width and a is the radiation damping width for absorption. The frequency ν' as seen from the moving atom is related to the frequency ν as seen from the reference frame fixed in space by

$$h\nu'=h\nu\left(1-\frac{V+v\cos\Theta}{c}\right).$$

Hence the energy absorbed per unit volume per unit time in the frequency range ν' to $\nu'+d\nu'$ is

$$\int_\Omega\frac{d\Omega}{4\pi}h\nu' B_{12}(\nu')I(\nu)d\nu\int_{v=0}^\infty 2\sqrt{\frac{M}{2\pi kT}}e^{-Mv^2/2kT}dv$$

$$=N_1B_{12}\frac{a}{\pi}\frac{1}{\sqrt{\pi}(\Delta\nu_D)}e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2}h\nu d\nu\times\left\{\frac{1-\dfrac{V}{c}}{a^2+\left[\nu-\nu_0\left(1-\dfrac{V}{c}\right)\right]^2}\right.$$

$$\left.+\frac{\nu_0}{2c^2}\left(\frac{2kT}{M}\right)\cdot\frac{\left\{\left[\nu-\nu_0\left(1-\dfrac{V}{c}\right)\right]^2-\dfrac{a^2}{3}\right\}\left\{4\nu-3\nu_0\left(1-\dfrac{V}{c}\right)\right\}}{\left\{a^2+\left[\nu-\nu_0\left(1-\dfrac{V}{c}\right)\right]^2\right\}^3}+\cdots\right\}.$$

The first term is equal to the amount of the absorbed energy when the thermal motion is not taken into account. The second term represents the correction due to the thermal motion of the atoms. The effect of the thermal motion is in the sense of increasing the absorbed energy and hence the radiation pressure. The effect of the systematic motion, when we neglect the effect of the thermal motion, is expressed by

$\Psi(\nu; r)I\,d\nu$

$$=N_1B_{12}I\cdot\left(1-\frac{V}{c}\right)\frac{2h\nu_0}{\sqrt{\pi}\,(\Delta\nu_D)}\,e^{a^2/(\Delta\nu_D)^2}\cdot Erfc\left(\frac{a}{\Delta\nu_D}\right)$$

$$-N_1B_{12}I\cdot 2\frac{V}{c}h\nu_0\frac{1}{\sqrt{\pi}\,(\Delta\nu_D)}\,\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{(a^2+x^2)^2}e^{-x^2/(\Delta\nu_D)^2}dx+\cdots,$$

where

$$Erfc(x)=\int_x^{\infty}e^{-t^2}dt=\frac{\sqrt{\pi}}{2}-\int_0^x e^{-t^2}dt.$$

This effect is the shift of the line center in the sense of decreasing the radiation pressure, by more than the amount shown by the factor $\left(1-\frac{V}{c}\right)$ of the first line.

3. *Line Emission.* Consider an atom moving with velocity $V+v\cos\Theta$ in the z-direction. Suppose that the probability of emission for the line as seen from the moving atom is

$$A_{21}(\nu')d\nu'=\frac{b}{\pi}\,\frac{1}{b^2+(\nu'-\nu_0')^2}d\nu'$$

and the number of atoms in the second quantum state per unit volume is

$$N_2(v)=2\sqrt{\frac{M}{2\pi kT}}\,e^{-Mv^2/2kT}dv\,\frac{d\Omega}{4\pi}.$$

Then the emitted energy in the frequency range ν to $\nu+d\nu$ per unit volume per unit time as seen from the fixed frame is

$$\begin{aligned}E(\nu;r)d\nu&\\=N_2(v)dv\cdot A_{21}(\nu')h\nu'd\nu'&=N_2A_{21}\cdot\frac{b}{\pi}h\nu d\nu\left\{\frac{1-\frac{V}{c}}{b^2+\left[\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2}\right.\\&\left.+\frac{\nu_0}{2c^2}\left(\frac{2kT}{M}\right)\frac{\left\{\left[\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2-\frac{b^2}{3}\right\}\left\{4\nu-3\nu_0\left(1-\frac{V}{c}\right)\right\}}{\left\{b^2+\left[\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2\right\}^3}+\cdots\right\}.\end{aligned}$$

The first term gives the ordinary emission formula with the shift of the line center due to the Doppler effect and the factor $\left(1-\frac{V}{c}\right)$ for correcting the energy of a light quantum $h\nu$. The second term represents the effect of the thermal motion.

4. *Transfer of the Line Radiation.* Denote by r the distance from the center of the exciting star to a point in the cloud and θ is the angle between the radius vector and the direction of the radiation and write

$$\omega(\nu;r)=\frac{1}{4\pi}\Psi(\nu;r),\qquad Q(\nu;r)=\frac{E(\nu;r)}{\Psi(\nu;r)}.$$

Then the equation of the transfer as seen from the fixed frame is (Hagihara 1938, Hagihara and Hatanaka 1941)

$$\cos\theta\,\frac{\partial I(\nu;\,r)}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial I(\nu;\,r)}{\partial\theta}=-\omega(\nu;\,r)I(\nu;\,r)+\omega(\nu;\,r)Q(\nu;\,r).$$

Further write

$$\omega(\nu_0;\,r)dr=\frac{1}{4\pi}\,\Psi(\nu_0;\,r)dr=dt,$$

$$\frac{\omega(\nu;\,r)}{\omega(\nu_0;\,r)}=\frac{1}{\sqrt{3}}\,\lambda(\nu;\,r),$$

and suppose that $t=0$ at the inner boundary of the cloud where $r=r_0$, then we get after the way of Chandrasekhar (Chandrasekhar 1934, Hagihara 1943) with Eddington's approximation

$$\frac{1}{r^2}\,\frac{d}{dt}\left[\,r^2\,\frac{dJ(\nu;\,r)}{dt}\,\right]=\lambda^2(\nu;\,r)J(\nu;\,r)-\frac{1}{3}\,\lambda^2(\nu;\,r)Q(\nu;\,r),$$

where

$$J(\nu;\,r)=\int_0^{\pi}I(\nu;\,r)\,\frac{d\omega}{4\pi}\,.$$

If we neglect the thermal motion of the atoms, then we have

$$\omega(\nu;\,r)=\frac{1}{4\pi}\,N_1B_{12}\cdot\frac{a}{\pi}\,\frac{1}{\sqrt{\pi}\,(\Delta\nu_D)}\left(1-\frac{V}{c}\right)\frac{h\nu}{a^2+\left[\,\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2}\,e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2},$$

$$\omega(\nu_0;\,r)=\frac{1}{4\pi}\,N_1B_{12}\cdot\frac{a}{\pi}\,\frac{1}{\sqrt{\pi}\,(\Delta\nu_D)}\left(1-\frac{V}{c}\right)\frac{h\nu}{a^2},$$

$$\lambda(\nu;\,r)=\sqrt{3}\,\frac{a^2}{a^2+\left[\,\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2}\,e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2},$$

$$Q(\nu;\,r)=\frac{N_1B_{12}a}{N_2A_{21}b}\cdot\frac{1}{\sqrt{\pi}\,(\Delta\nu_D)}\,\frac{b^2+\left[\,\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2}{a^2+\left[\,\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2}\,e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2},$$

$$dt=\frac{1}{4\pi}\,N_1B_{12}\cdot\frac{a}{\pi}\,\frac{1}{\sqrt{\pi}\,(\Delta\nu_D)}\left(1-\frac{V}{c}\right)\frac{h\nu}{a^2}\,dr.$$

5. *Resonance Line.* If we suppose that V, T, N_1 and N_2 are one-valued functions of r, then r is a function of the optical depth t only and hence V, T, N_1 and N_2 are functions of t only. $Q(\nu;\,r)$ is a function of t through V, T and $N_2\!:N_1$, and $\lambda(\nu;\,r)$ may be a function of t through V and $(\Delta\nu_D)$ which is a function of T. Then the equation of transfer is a Sturm-Liouville's boundary value problem (Hagihara 1943) for the differential equation

$$\frac{1}{r^2}\,\frac{d}{dt}\left[\,r^2\,\frac{dJ(\nu;\,r)}{dt}\,\right]=\lambda^2(\nu;\,r)J(\nu;\,r)-\frac{1}{3}\,\lambda^2(\nu;\,r)Q(\nu;\,r).$$

Denote the value of t for the outer boundary of the cloud by t_1 and the inner boundary by $t=0$. By the relation in Eddington's approximation we can express the boundary conditions at the outer boundary by

$$\frac{3}{2}J(\nu; r)+\frac{1}{\lambda(\nu; r)}\frac{dJ(\nu; r)}{dt}=0 \quad \text{at} \quad t=t_1$$

and at the inner boundary by

$$J(\nu; r)=\frac{1}{4}S(\nu) \qquad \text{at} \quad t=0,$$

where $\pi S(\nu)d\nu$ denotes the intensity of the stellar radiation passing through a unit area of the inner boundary of the cloud, which is moving towards the z-direction with velocity V, per unit time in the frequency range ν to $\nu+d\nu$. S should be multiplied by $\left(1-\frac{V}{c}\right)$ if S is taken as seen from the fixed frame.

Put

$$r^2dt=dx$$

and suppose that $x=x_1$ at $t=t_1$ and $x=0$ at $t=0$, then the differential equation takes the form

$$\frac{d}{dx}\left[r^4\frac{d}{dx}J(\nu; x)\right]-\lambda^2(\nu; x)J(\nu; x)=-\frac{1}{3}\lambda^2(\nu; x)Q(\nu; x),$$

with the boundary conditions

$$\frac{3}{2}J(\nu; x_1)+\frac{r^2(x_1)}{\lambda(\nu; x_1)}\frac{dJ(\nu; x_1)}{dx_1}=0 \qquad \text{at} \quad x=x_1,$$

and

$$J(\nu; 0)=\frac{1}{4}S(\nu) \qquad \text{at} \quad x=0.$$

Consider the Green function $G(\nu; x, \xi)$ satisfying these boundary conditions and the relation (Hagihara 1943)

$$\frac{d}{dx}\left[r^4\frac{dG(\nu; x, \xi)}{dx}\right]-\lambda^2(\nu; x)G(\nu; x, \xi)=0,$$

and

$$G(\nu; x, \xi)=G(\nu; \xi, x),$$

$$\frac{\partial}{\partial\xi}G(\nu; \xi, \xi+0)-\frac{\partial}{\partial\xi}G(\nu; \xi, \xi-0)=-\frac{1}{r^4(\xi)}.$$

Then the solution of the boundary value problem is

$$J(\nu; x)=\frac{1}{3}\lambda^2(\nu; x)\int_0^{x_1}G(\nu; x, \xi)Q(\nu; \xi)d\xi.$$

The radiation pressure K is given by Eddington's approximation

$$K=\int_0^\infty d\nu\int_{r_0}^{r_1}\frac{4\pi}{3c}J(\nu; r)dr$$

$$=\int_0^\infty d\nu\int_0^{x_1}\frac{4\pi}{9c}\lambda^2(\lambda; x)\,dx\int_0^{x_1}G(\nu; x, \xi)Q(\nu; \xi)d\xi.$$

6. *Thin Cloud.* If the cloud is sufficiently thin, then we can assume that r is constant, throughout the cloud and also that T, N_1 and N_2 are constant, and hence that $\lambda(\nu; r)$ is constant throughout the cloud. The solution of the differential equation

$$\frac{1}{r^2}\frac{d}{dt}\left(r^2\frac{dJ}{dt}\right)-\lambda^2 J=-\frac{1}{3}\lambda^2 Q$$

is

$$J=Ae^{\lambda t}+Be^{-\lambda t}+\frac{1}{3}Q$$

with integration-constants A and B, which should be determined by the boundary conditions. We get

$$A=-\frac{\left(\frac{1}{4}S-\frac{1}{3}Q\right)e^{-\lambda t_1}+Q}{5e^{\lambda t_1}-e^{-\lambda t_1}},\quad B=\frac{5\left(\frac{1}{4}S-\frac{1}{3}Q\right)e^{\lambda t_1}+Q}{5e^{\lambda t_1}-e^{-\lambda t_1}}$$

and

$$\lambda(\nu)=\sqrt{3}\,\frac{a^2}{a^2+\left[\nu-\nu_0\left(1-\frac{V}{c}\right)\right]^2}e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2}.$$

The radiation pressure is

$$K=\frac{4\pi}{3c}\int_0^\infty\frac{1}{\lambda(\nu)}\left[A(e^{\lambda(\nu)t_1}-1)-B(e^{-\lambda(\nu)t_1}-1)+\frac{1}{3}\lambda(\nu)Q(\nu)t_1\right]d\nu.$$

Assume that t_1 is small, then we obtain

$$\begin{aligned}\frac{3c}{4\pi}=&K\int_0^\infty\frac{1}{4}S(\nu)d\nu\\&-(1-t_1)\frac{1}{3}\frac{N_1B_{12}a}{N_2A_{21}b}\left\{1+\frac{2(b^2-a^2)}{a(\Delta\nu_D)}e^{a^2/(\Delta\nu_D)^2}\cdot Erfc\left(\frac{a}{\Delta\nu_D}\right)\right.\\&\left.+\frac{2V}{c}\frac{(b^2-a^2)}{\sqrt{\pi}\,(\Delta\nu_D)}\int_{-\infty}^\infty\frac{x^2}{(a^2+x^2)^2}e^{-x^2/(\Delta\nu_D)^2}dx+\cdots\right\}.\end{aligned}$$

The second term in the bracket represents the effect of the emission by the atoms in the cloud. The third term in the bracket shows the effect of the systematic motion, besides the factor $\left(1-\frac{V}{c}\right)$ to the incident radiation $\pi S(\nu)$, which has been supposed to represent the incident flux per unit area per unit time to the inner boundary of the cloud with respect to the moving cloud. The term expresses the loss in radiation pressure due to the shift of the line center.

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