9. On Summability $[c, k]$ and Summability $[R, k]$ of Laplace Series

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1. If $f(\theta, \phi)$ be a function defined for the range $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, the Laplace series associated with this function on the sphere S is

$$f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+1/2) \iint_S f(\theta', \phi') P_n(\cos \gamma) \sin \theta' d\theta' d\phi', \tag{1.1}$$

where

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'),$$

and $P_n(x)$ denotes the n-th Legendre polynomial.

The generalized mean value of $f(\theta, \phi)$ is given by

$$f(\gamma) = \frac{1}{2\pi \sin \gamma} \int_{C\gamma} f(\theta', \phi')\, dS', \tag{1.2}$$

where the integral is taken along the small circle whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is γ.

The series

$$\sum_{n=0}^{\infty} u_n \tag{1.3}$$

is said to be strongly summable (c, k) or summable $[c, k]$ to the sum S, if

$$\sum_{\nu=0}^{n} | s_\nu^{(k-1)} - s | = 0(n), \tag{1.4}$$

where $s_\nu^{(k-1)}$ denotes the ν-th cesaro mean of order $(k-1)$ of the series (1.3).

Again, we say that the series (1.3) is strongly summable (R, k) or summable $[R, k]$ to the sum S, if

$$\sum_{\nu=0}^{n} \frac{| s_\nu^{(k-1)} - s |}{\nu+1} = 0 (\log n). \tag{1.5}$$

The object of this paper is to obtain some new results for the series (1.1) on its $[c, k]$ and $[R, k]$ summability.

We prove the following theorems:

Theorem 1: *If*

$$\varphi(t) \equiv \int_t^{\delta} \frac{|\phi(\gamma)|}{\gamma} d\gamma = 0\left[t\left(\log \frac{1}{t}\right)^{\alpha}\right], \quad (\alpha > 0) \tag{1.6}$$

as $t \to 0$, $(0 < \delta \leq \pi)$, then

$$\sum_{\nu=0}^{n} | \sigma_\nu^{(k)}(\gamma) - \sigma | = 0[n(\log n)^{\alpha}], \quad (0 \leq k \leq 1), \tag{1.7}$$

where

$$\phi(\gamma)=[f(\gamma)-A]\sin\gamma,$$

and $\sigma_\nu^{(k)}(\gamma)$ *denotes the* ν*-th cesaro mean of order* k *of the series* (1.1).

Theorem 2: *If*

$$\varphi(t)=0\left[t\left(\log\frac{1}{t}\right)^{1+\alpha}\right], \tag{1.8}$$

then

$$\sum_{\nu=0}^{n}\frac{|\sigma_\nu^{(k)}(\gamma)-\sigma|}{\nu+1}=0[(\log n)^{1+\alpha}]. \tag{1.9}$$

2. *Proof theorem 1.* The ν-th cesaro mean of order k of the series (1.1) is given by

$$\sigma_\nu^{(k)}(\gamma)=\frac{1}{4\pi}\int_0^\pi f(\gamma)\sin\gamma\cdot S_\nu^{(k)}(\gamma)d\gamma,$$

where $S_\nu^{(k)}(\gamma)$ denotes the ν-th cesaro mean of order k of the series

$$\sum_{n=0}^{\infty}(2n+1)P_n(\cos\gamma). \tag{2.1}$$

It is known that (see Hobson, E. W.: Spherical and Ellipsoidal Harmonics, 1931, p. 355):

$$|S_\nu^{(k)}(\gamma)|\leq\begin{cases}\dfrac{B_1}{\nu^{k-1/2}}\cdot\dfrac{1}{\gamma^{1+k}}\sin^{-1/2}\gamma, & (0<\gamma\leq\pi);\\ B_2\nu^2, & (0\leq\gamma\leq\pi),\\ B_1 \text{ and } B_2 \text{ are independent of } \nu \text{ and } \gamma.\end{cases} \tag{2.2}$$

Thus, we have

$$\begin{aligned}\sum_{\nu=0}^{n}|\sigma_\nu^{(k)}(\gamma)-\sigma| &=\frac{1}{4\pi}\sum_{\nu=0}^{n}\int_0^\pi\phi(\gamma)S_\nu^{(k)}(\gamma)d\gamma+0(n)\\ &=\frac{1}{4\pi}\sum_{\nu=0}^{n}\left[\int_0^{1/n}+\int_{1/n}^{\pi}\right]+0(n).\\ &=J_1+J_2+0(n),\quad\text{say.}\end{aligned} \tag{2.3}$$

Now using (2.2) we have $J_1=0(n^3)\int_0^{1/n}|\phi(\gamma)|d\gamma$.

But we observe that, if

$$\varphi(t)=0\left\{t\left(\log\frac{1}{t}\right)^{\alpha}\right\},\quad \alpha>0$$

then

$$\Phi(t)=\int_0^t|\phi(u)|du=0\left\{t^2\left(\log\frac{1}{t}\right)^{\alpha}\right\}.$$

For, $\Phi(t)=\int_0^t|\phi(u)|du$

$$\begin{aligned}&=\int_0^t-u\varphi'(u)du,\quad\text{where } \varphi'(u)=\frac{d}{du}\{\varphi(u)\}\\ &=0\left[t^2\left(\log\frac{1}{t}\right)^{\alpha}\right]+0\left[\int_0^t u\left(\log\frac{1}{u}\right)^{\alpha}du\right]=0\left[t^2\left(\log\frac{1}{t}\right)^{\alpha}\right].\end{aligned}$$

Therefore, we have

$$J_1 = 0(n^3)\left\{0\left[\gamma^2\left(\log\frac{1}{\gamma}\right)^{\alpha}\right]_0^{1/n}\right\}$$

$$= 0[n(\log n)^{\alpha}]. \tag{2.4}$$

Again

$$J_2 \leq \frac{1}{4\pi}\sum_{\nu=0}^{n}\int_{1/n}^{\pi}\frac{|\phi(\gamma)|}{\gamma^{1+k}}\cdot\nu^{1/2-k}\cdot\frac{B_1}{\sin^{1/2}\gamma}\,d\gamma$$

$$= 0(n^2)\int_{1/n}^{\pi}\frac{|\phi(\gamma)|}{\gamma}\,d\gamma$$

$$= 0[n(\log n)^{\alpha}]. \tag{2.5}$$

Thus the theorem follows from (2.3), (2.4), and (2.5).

3. *Proof of theorem 2.* We have

$$\sum_{\nu=0}^{n}\frac{|\sigma_{\nu}^{(k)}(\gamma)-\sigma|}{\nu+1} = \frac{1}{4\pi}\sum_{\nu=0}^{n}\int_0^{\pi}|\phi(\gamma)|\cdot\frac{S_{\nu}^{(k)}(\gamma)}{\nu+1} + \sum_{\nu=0}^{n}\frac{0(1)}{\nu+1}$$

$$= \frac{1}{4\pi}\sum_{\nu=0}^{n}\left[\int_0^{1/n} + \int_{1/n}^{\pi}\right] + 0(\log n)$$

$$= I_1 + I_2 + 0\,(\log n), \tag{3.1}$$

say. Now, treating I_1 in the same manner as J_1, we obtain

$$I_1 = 0(n^2)\int_0^{1/n}|\phi(\gamma)|\,d\gamma$$

$$= 0[(\log n)^{1+\alpha}] \tag{3.2}$$

using the hypothesis (1.8). Also,

$$I_2 = 0(n)\int_{1/n}^{\pi}\frac{|\phi(\gamma)|}{\gamma}\,d\gamma$$

$$= 0[(\log n)^{1+\alpha}]. \tag{3.3}$$

Thus in view of the relations (3.1), (3.2), and (3.3), the theorem is proved.

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