1. Existence of Pseudo-Analytic Differentials on Riemann Surfaces. I

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In this paper, we shall prove the existence theorems for (F, G)-pseudo-analytic differentials in the sence of Bers (Bers, L., [1], [2]) on arbitrary Riemann surfaces, under the condition:

(1) $$-i\bar{F}G>0, \quad M\geqq|F|+|G|\geqq M^{-1}>0.$$

We consider the differential $\omega=\sqrt{\sigma}\,du$, u being locally a solution of the partial differential equation

(2) $$(\sigma u_x)_x+(\sigma u_y)_y=0,$$

where σ being a positive function on Riemann surface. A generalization of Weyl's lemma for this differential is proved, and the method of orthogonal projection is used.

I. $[a, b]$-analytic functions and differentials. 1. Let Ω be a domain of z-plane. A subdomain Ω_0 of Ω is called the compact subdomain of Ω, if $\bar{\Omega}_0\subset\Omega$ and denoted by $\Omega_0\Subset\Omega$. The class of functions continuous on Ω (or, which have continuous partial derivatives up to the n-th order) is denoted by $C(\Omega)$ (or $C^n(\Omega)$). The class of functions whose n-th order partial derivatives are all uniformly α-Hölder continuous ($0<\alpha<1$) in Ω, is denoted by $C^{n+\alpha}(\Omega)$. The class of functions of $C(\Omega)(C^n(\Omega), C^{n+\alpha}(\Omega))$ which have compact carrier in Ω is denoted by $C_0(\Omega)(C_0^n(\Omega), C_0^{n+\alpha}(\Omega))$. The class of functions square summable on every compact subdomain of Ω is denoted by $\mathfrak{L}^2(\Omega)$.

Definition 1.1. *A function $f(z)$ of $\mathfrak{L}^2(\Omega)$ is said to be in the class $\mathfrak{D}_{\bar{z}}(\Omega)$, if there exists a function $g(z)\in\mathfrak{L}^2(\Omega)$ such that, for every function $\phi(z)$ of $C_0^2(\Omega)$,*

(1.1) $$\iint_\Omega\{f(z)\phi_{\bar{z}}(z)+g(z)\phi(z)\}dxdy=0$$

holds. In this case, we write $g(z)=f_{\bar{z}}(z)$.

We note that the condition (1.1) is replaced by

(1.1)′ $$Re\iint_\Omega\{f(z)\phi_{\bar{z}}(z)+g(z)\phi(z)\}dxdy=0.$$

Lemma 1.1. *If $f(z)\in\mathfrak{D}_{\bar{z}}(\Omega)$ and $f_{\bar{z}}(z)=0$ a.e. in Ω, then $f(z)$ is analytic in Ω.*

Proof. Let Ω_0 be any compact subdomain of Ω. Let $L^2(\Omega_0)$ be the Hilbert space of the functions square summable on Ω_0, $E(\Omega_0)$ be the closed subspace of $L^2(\Omega_0)$ spanned by the functions ϕ_z with $\phi\in C_0^2(\Omega)$. The orthogonal complement of $E(\Omega_0)$ in $L^2(\Omega_0)$ is denoted by $A(\Omega_0)$. We shall prove that all the functions of $A(\Omega_0)$ are analytic. If $f(z)$

belongs to $A(\Omega_0)\cap C^1(\Omega_0)$, it is analytic. Let J_ε denote the molifier (K. O. Friedrichs [3]). If $f(z)$ is any function of $A(\Omega_0)$, then

$$(J_\varepsilon f, \phi_z)=(f, J_\varepsilon\phi_z)=(f, (J_\varepsilon\phi)_z)=0$$

holds for every $\phi(z)\in C_0^2(\Omega_0)$, and for sufficiently small ε. Therefore, $J_\varepsilon f\in A(\Omega_0)$, and, since $J_\varepsilon f\in C^2(\Omega_0)$, it is analytic. On the other hand, $J_\varepsilon f$ converges to $f(z)$ in $L^2(\Omega_0)$ and hence uniformly in every compact subdomain of Ω_0. This implies the analyticity of $f(z)$. If $f(z)\in\mathfrak{D}_{\bar z}(\Omega)$ and $f_{\bar z}(z)=0$, then $f\in L^2(\Omega_0)$ and

$$(f, \bar\phi_z)=\iint_\Omega f\phi_{\bar z}dxdy=0$$

holds for every $\phi(z)\in C_0^2(\Omega)$ and hence we have $f(z)\in A(\Omega_0)$ which proves the lemma.

Lemma 1.2. *Let Ω be a bounded domain and $\rho(z)$ be a bounded measurable function on Ω. Set*

$$\sigma(z)=-\frac{1}{\pi}\iint_\Omega \frac{\rho(\zeta)}{\zeta-z}d\xi d\eta,$$

then we have

(1) *$\sigma(z)$ is in $C^\alpha(\Omega)$, and is bounded in Ω.*

(2) *$\sigma(z)$ is in $\mathfrak{D}_{\bar z}(\Omega)$, and $\sigma_{\bar z}(z)=\rho(z)$ a.e. in Ω.*

(3) *If $\rho(z)\in C^\alpha(\Omega)$, then $\sigma(z)\in C^{1+\alpha}(\Omega)$.*

This is the well-known result.

2. Let Ω be a bounded domain and $a(z)$, $b(z)$ be functions of $C^\alpha(\Omega)$.

Definition 1.2. *A function $f(z)$ of $C^1(\Omega)$ is called an $[a, b]$-analytic function if it satisfies the equation*

$$f_{\bar z}=af+b\bar f \quad a.e.\ in\ \Omega. \tag{1.2}$$

Lemma 1.3. *If $f(z)$ is a bounded function of $\mathfrak{D}_{\bar z}(\Omega)$ and satisfies (1.2) a.e. in Ω, then $f(z)$ is $[a, b]$-analytic.*

Proof. Consider the function

$$\varphi(z)=f(z)+\frac{1}{\pi}\iint_\Omega \frac{a(\zeta)f(\zeta)+b(\zeta)\overline{f(\zeta)}}{\zeta-z}d\xi d\eta. \tag{1.3}$$

Since $af+b\bar f$ is bounded, the integral of the right member is in $C^\alpha(\Omega)\cap\mathfrak{D}_{\bar z}(\Omega)$. We have $\varphi_{\bar z}(z)=0$ a.e. in Ω. By Lemma 1.1, $\varphi(z)$ is analytic. Therefore, we have $f(z)\in C^\alpha(\Omega)$, and hence $af+b\bar f$ is in $C^\alpha(\Omega)$, and we have consequently $f(z)\in C^{1+\alpha}(\Omega)$. This proves the lemma. (This proof contains the result that the $[a, b]$-analytic function belongs to $C^{1+\alpha}(\Omega)$.)

Lemma 1.4. (*Similarity principle.*) *If $f(z)\in\mathfrak{D}_{\bar z}(\Omega)$ and satisfies (1.2) a.e. in Ω, then there exists an analytic function $\varphi(z)$ similar to $f(z)$: that is, there exists a function $S(z)$ such that*

$$0<k^{-1}\leqq|S(z)|\leqq k$$

for some constant k, and such that

(1.4) $$\varphi(z)=S(z)f(z).$$

Proof. Let E be the set of points of Ω at which $f(z)=0$. Set

$$\rho(z)=\begin{cases} a(z)+b(z)\overline{f(z)}/f(z) & \text{in } \Omega-E. \\ a(z)+b(z) & \text{in } E. \end{cases}$$

Then, $\rho(z)$ is a bounded measurable function in Ω. Setting

(1.5) $$\sigma(z)=\frac{1}{\pi}\iint_{\Omega}\frac{\rho(\zeta)}{\zeta-z}d\xi d\eta,$$

we have $\sigma_{\bar z}(z)=-\rho(z)$ by Lemma 1.2. We set $\varphi(z)=S(z)f(z)$ with $S(z)=e^{\sigma(z)}$. Then, we have in $\Omega-E$, $\varphi_{\bar z}(z)=S(z)\{f_{\bar z}(z)-\rho(z)f(z)\}=S(z)\{f_{\bar z}-af-b\bar f\}=0$ and in E, $\varphi_{\bar z}(z)=S(z)\{f_{\bar z}(z)-\rho(z)f(z)\}=S(z)\{f_{\bar z}-af-bf\}=S\{f_{\bar z}-af-b\bar f\}=0$. Thus, we have $\varphi_{\bar z}(z)=0$ a.e. in Ω.

Lemma 1.5. *If $f(z)\in\mathfrak{D}_{\bar z}(\Omega)\cap L^2(\Omega)$ and satisfies* (1.2) *a.e. in Ω, then we have, in every compact subdomain Ω_0 of Ω,*

(1.6) $$|f(z)|\leqq k_0\|f\|_\Omega,$$

where k_0 is a constant depending to Ω_0.

Proof. Let δ be the distance between Ω_0 and $\partial\Omega$. We consider an arbitrary point $z_0\in\Omega_0$ and the disk $K:|z-z_0|\leqq\frac{\delta}{2}$. Define the analytic function $\varphi(z)$ of previous lemma. Then we have

$$\begin{aligned}|f(z_0)|^2&\leqq k^2|\varphi(z_0)|^2\\ &\leqq\frac{4k^2}{\pi\delta^2}\iint_K|\varphi(z)|^2dxdy\\ &\leqq\frac{4k^4}{\pi\delta^2}\iint_K|f(z)|^2dxdy\leqq k_0^2\|f\|_\Omega^2\end{aligned}$$

with $k_0=2k^2/(\sqrt{\pi}\,\delta)$.

If $f(z)\in\mathfrak{D}_{\bar z}(\Omega)$ and satisfies (1.2) a.e. in Ω, then for any compact subdomain Ω_0 of Ω, $f(z)\in L^2(\Omega_0)$ and hence $f(z)$ is bounded on every compact subdomain of Ω. Thus, from Lemma 1.3, we have

Theorem 1.1. *If $f(z)\in\mathfrak{D}_{\bar z}(\Omega)$ and satisfies* (1.2) *a.e. in Ω, then $f(z)$ is $[a,b]$-analytic in Ω.*

3. Let R be an arbitrary Riemann surface, and $C, C^n,\cdots$ etc. be the classes of functions which have the corresponding properties in every neighborhood. Let $a(z)d\bar z$, $b(z)dz$ be differentials of C^α.

Definition 1.3. *A differential $\varphi=fdz$ is called an $[a,b]$-analytic differential if $\varphi\in C^1$ and satisfies the equation*

(1.7) $$f_{\bar z}=af+b\bar f.$$

We consider the real Hilbert space $\boldsymbol{L}^2$ of pure differentials square summable on R. The inner product is defined by

(1.8) $$(\varphi,\varphi')=Re\iint_R\varphi\wedge*\bar\varphi',\quad \varphi,\varphi'\in\boldsymbol{L}^2.$$

We also consider the subspace

$\boldsymbol{E}$=closure of $\{\boldsymbol{D}\phi=(\phi_z+\bar{a}\phi+b\bar{\phi})dz;\ \phi\in C_0^2\}$ in $\boldsymbol{L}^2$.

The orthogonal complement of $\boldsymbol{E}$ in $\boldsymbol{L}^2$ is denoted by $\boldsymbol{A}$.

Theorem 1.2. *$\boldsymbol{A}$ is the space of $[a,b]$-analytic differentials in $\boldsymbol{L}^2$.*

Proof. Let $\varphi=fdz$ be in $\boldsymbol{L}^2$, then for every $\phi\in C_0^2$, we have

$$(\varphi,\boldsymbol{D}\bar{\phi})=Re\iint_R fdz\wedge i\{\overline{\bar{\phi}_z+\bar{a}\bar{\phi}+b\phi}\}d\bar{z} \tag{1.9}$$

$$=2Re\iint_R\{f\phi_{\bar{z}}+(af+b\bar{f})\phi\}dxdy.$$

If φ is $[a,b]$-analytic and is in $\boldsymbol{L}^2$, then the right member vanishes and therefore $\varphi\in\boldsymbol{A}$. Conversely, if $\varphi\in\boldsymbol{A}$, then for every $\phi\in C_0^2$, we have

$$Re\iint_R\{f\phi_{\bar{z}}+(af+b\bar{f})\phi\}dxdy=0.$$

Therefore φ is in $\mathfrak{D}_{\bar{z}}$, and satisfies (1.7). By Theorem 1.1, φ is $[a,b]$-analytic.

II. σ-harmonic differentials. 1. In this chapter, we consider a generalization of harmonic differentials. Let R be an arbitrary Riemann surface and $\sigma(p)$ be a function of $C^{1+\alpha}$, such that $M\geqq\sigma\geqq M^{-1}>0$ on R.

We define the differential operators D, D_1, and D_2, as follows:

(2.1) $Du=\sqrt{\sigma}\,du$ for a real function $u(p)$ of C^1.

(2.2) $D_1\omega=d\left(\frac{1}{\sqrt{\sigma}}\omega\right)$ for a real differential $\omega\in C^1$.

$D_2\omega=d(\sqrt{\sigma}\,\omega)$

Definition 2.1. *A real differential $\omega\in C^1$ is called σ-harmonic differential if $D_1\omega=0$ and $D_2*\omega=0$ hold.*

The condition $D_1\omega=0$ implies that ω is written as $\omega=Du$ locally, and if, moreover, $D_2*\omega=0$, then $u(z)$ satisfies the equation

$$(\sigma u_x)_x+(\sigma u_y)_y=0. \tag{2.3}$$

Definition 2.2. *A real function $u(p)$ defined on a domain $\Omega\subset R$ is called σ-harmonic function on Ω, if it satisfies (2.3) in Ω.*

2. Let L^2 be the Hilbert space of real differentials square summable on R. Consider the subspaces

$$\begin{aligned} E\ &=\text{closure of } \{D\phi;\ \phi\in C_0^2\} \quad \text{in } L^2\\ E^*&=\text{closure of } \left\{\frac{1}{\sigma}*D\phi;\ \phi\in C_0^2\right\} \quad \text{in } L^2. \end{aligned} \tag{2.4}$$

Lemma 2.1. *A differential ω of $C^1\cap L^2$ is σ-harmonic if and only if $\omega\perp E$ and $\omega\perp E^*$.*

Lemma 2.2. *The space E and E^* are mutually orthogonal.*

The statements are easily seen by the relations:

$$(\omega,D\phi)=\iint_R\omega\wedge\sqrt{\sigma}*d\phi=\iint_R\phi D_2*\omega$$

$$\left(\omega, \frac{1}{\sigma} * D\phi\right) = -\iint_R \omega \wedge \frac{1}{\sqrt{\sigma}} d\phi = \iint_R \phi D_1 \omega$$

$$\left(D\phi, \frac{1}{\sigma} * D\phi'\right) = -\iint_R \sqrt{\sigma}\, d\phi \wedge \frac{1}{\sqrt{\sigma}} d\phi' = -\iint_R d\phi \wedge d\phi' = 0.$$

The orthogonal complement of $E \oplus E^*$ in L^2 is denoted by H.

Lemma 2.3. (*Generalization of Weyl's lemma.*) *All the differentials of H are in $C^{1+\alpha}$, and therefore H is the space of σ-harmonic differentials in L^2.*

Proof. We set

$$a = \frac{\sigma_{\bar{z}}}{2\sigma} \qquad b = \frac{-\sigma_z}{2\sigma}. \tag{2.5}$$

Then the differentials $ad\bar{z}$ and bdz belong to C^α. For eveay $\phi = \phi' + i\phi'' \in C_0^2$, we have

$$\begin{aligned}(\sqrt{\sigma}(\omega + i*\omega), \boldsymbol{D}\bar{\phi}) &= Re\iint_R \sqrt{\sigma}(\omega + i*\omega) \wedge i(\phi_{\bar{z}} + a\phi + \bar{b}\bar{\phi})d\bar{z} \\ &= Re\iint_R \sqrt{\sigma}(\omega + i*\omega) \wedge \left\{ i\phi'_{\bar{z}} d\bar{z} - \frac{1}{\sigma}(\sigma\phi'')_{\bar{z}} d\bar{z} \right\} \\ &= \iint_R \sqrt{\sigma}\,\omega \wedge *d\phi' - \iint_R \omega \wedge \frac{1}{\sqrt{\sigma}} d(\sigma\phi''). \\ &= (\omega, D\phi') - \left(\omega, \frac{1}{\sigma} D*(\sigma\phi'')\right).\end{aligned}$$

Since ϕ' and $\sigma\phi''$ are in C_0^2, the right member vanishes. This implies that $\sqrt{\sigma}(\omega + i*\omega)$ belongs to $\boldsymbol{A}$, and hence $\omega \in C^{1+\alpha}$. Thus we have

Theorem 2.1. *If ω is a differential of L^2, then ω is decomposed into*

$$\omega = \omega_h + \omega_1 + \omega_2 \tag{2.6}$$

where ω_h is σ-harmonic, $\omega_1 \in E$ and $\omega_2 \in E^*$.

3. To obtain the further results, we shall prove

Lemma 2.4. *If $\omega \in E \cap C^1$, then $\omega = Du$ for a function $u \in C^2$. If $\omega \in E^* \cap C^1$, then $\omega = Dv$ for a function $v \in C^2$.*

Proof. Suffice it to prove the first statement. Let γ be an arbitrary analytic closed curve on R, and G be a doubly connected domain containing γ as its separating curve and possessing the smooth boundary curves. The right and left subdomains of G are denoted by G^+ and G^- respectively. We can construct a function $f(p) \in C^2(G)$ by

$$f(p) = \begin{cases} 1 & \text{for} \quad p \in G^- \cup \gamma \\ 0 & \text{for} \quad p \in R - G, \end{cases}$$

and a differential $\eta \in C^1$ by

$$\eta = \begin{cases} df & \text{in} \quad G \\ 0 & \text{in} \quad R - G. \end{cases}$$

Since $\omega \in E \cap C^1$, we have

$$\left(\omega, \frac{1}{\sqrt{\sigma}}*\eta\right)=-\iint_R \omega\wedge\frac{1}{\sqrt{\sigma}}\eta=-\int_\gamma \frac{1}{\sqrt{\sigma}}\omega.$$

On the other hand, there is a sequence $\{\omega_n\}\subset E$ such that $\omega_n=D\phi_n$ with $\phi_n\in C_0^2$ and $\|\omega_n-\omega\|\to 0$ as $n\to\infty$. Since η is closed, we have $\left(\frac{1}{\sqrt{\sigma}}\omega_n, *\eta\right)=-\iint_R d\phi_n\wedge\eta=0$. Consequently we have $\int_\gamma \frac{1}{\sqrt{\sigma}}\omega=0$. It implies that $\frac{1}{\sqrt{\sigma}}\omega$ is exact and that $\omega=\sqrt{\sigma}\,du$ with $u\in C^2$.

Lemma 2.5. *If $\omega\in C^{1+\alpha}$, then locally $\omega=Du+\frac{1}{\sigma}*Dv$ with $u, v\in C^2$.*

Proof. Let V be any neighborhood and $|z|<1$ be the parametric disk corresponding to V. If $\omega=p(z)dx+q(z)dy$ in V, then the function $h(z)=\left(\frac{1}{\sqrt{\sigma}}q\right)_x-\left(\frac{1}{\sqrt{\sigma}}p\right)_y$ is in $C^\alpha(V)$. We consider the equation

$$\left(\frac{1}{\sigma}v_x\right)_x+\left(\frac{1}{\sigma}v_y\right)_y=h(z). \tag{2.7}$$

For sufficiently small $r<1$, we can find a solution $v(z)\in C^2$ in the disk $|z|<r$. (2.7) implies $D_1\left(\omega-\frac{1}{\sigma}*Dv\right)=0$, and hence, by the previous lemma, there is a function $u(z)$ in the neighborhood corresponding to $|z|<r$ such that $\omega-\frac{1}{\sigma}*Dv=Du$.

Theorem 2.2. *If $\omega\in L^2\cap C^{1+\alpha}$, then $\omega=\omega_h+Du+\frac{1}{\sigma}*Dv$ with $u, v\in C^2$ and $\omega_h\in H$.*

Proof. By Theorem 2.1, we have $\omega=\omega_h+\omega_1+\omega_2$ with $\omega_h\in H$, $\omega_1\in E$ and $\omega_2\in E^*$. By Lemma 2.5, in a small neighborhood of every point of R, we have $\omega=Du_0+\frac{1}{\sigma}*Dv_0$ with $u_0, v_0, \in C^2(V)$. Set in V,

$$\theta=\omega_h+\omega_1-Du_0=-\omega_2+\frac{1}{\sigma}*Dv_0. \tag{2.8}$$

For every $\phi\in C_0^2(V)$, we have $(\theta, D\phi)_V=0$ and $\left(\theta, \frac{1}{\sigma}D*\phi\right)_V=0$. Hence, by Lemma 2.3, θ is in $C^1(V)$. Since $\omega_1=\theta-\omega_h+Du_0$ and $\omega_2=\frac{1}{\sigma}*Dv_0-\theta$, we have $\omega_1, \omega_2\in C^1$, which prove the theorem.

4. We consider another decomposition. Define the subspace

$$\widetilde{E}=\text{closure of } \{Du; u\in C^2\} \text{ in } L^2. \tag{2.9}$$

Let $\widetilde{H}$ be the orthogonal complement of $\widetilde{E}$ in $E\oplus H$. We have

Theorem 2.3. *If $\omega\in L^2\cap C^{1+\alpha}$, then ω is decomposed into*

$$\omega=\omega_h+Du+\frac{1}{\sigma}*Dv \tag{2.10}$$

with $u, v\in C^2$ and $\omega_h\in H$, $Du\in\widetilde{E}$ and $\frac{1}{\sigma}*Dv\in E^*$.

(See References of the following article.)