38. A Theorem of Bari on the Completeness of Orthonormal Systems

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The purpose of the present note is to give an another proof of the following

THEOREM. *If* $\{\varphi_n\}$ *is a complete orthonormal system of a Hilbert space, and if* $\{\psi_n\}$ *is an another orthonormal system such as*

$$\sum_{n=1}^{\infty}||\varphi_n-\psi_n||^2<\infty, \tag{1}$$

then $\{\psi_n\}$ *is complete too.*

The theorem is established by Nina Bari in 1941, according to her obituary note.[1] K. Iséki, in a note [2] published in these Proceedings, summarized several extensions of her theorem due to several authors. Recently, G. Birkhoff and G.-C. Rota [1] reproduced the theorem in a connection with the Sturm-Liouville expansions.

In Birkhoff-Rota's proof, the following lemma plays a central role:

LEMMA. *Under the hypothesis of the theorem, if* m *is a natural number such as*

$$\sum_{n=m+1}^{\infty}||\varphi_n-\psi_n||^2<1, \tag{2}$$

then the sequence of vectors

$$\varphi_1, \varphi_2, \cdots, \varphi_m, \psi_{m+1}, \psi_{m+2}, \cdots \tag{3}$$

is complete in the sense that no non-zero vector is orthogonal to (3).

Birkhoff-Rota's proof of the lemma is a simple application of the Parseval relation. In the present note, we shall give an alternative proof basing on the invertibility of an operator U defined by

$$Ux=\sum_{n=1}^{m}\alpha_n\varphi_n+\sum_{n=m+1}^{\infty}\alpha_n\psi_n \quad \text{for} \quad x=\sum_{n=1}^{\infty}\alpha_n\varphi_n. \tag{4}$$

We can easily obtain that U is a bounded operator which satisfies

$$||I-U||^2\leqq||I-U||_2^2=\sum_{n=1}^{\infty}||(I-U)\varphi_n||^2=\sum_{n=m+1}^{\infty}||\varphi_n-\psi_n||^2<1,$$

since the uniform norm $||T||$ of an operator T is not greater than the Schmidt norm $||T||_2$ (e.g. [3]). Hence U has an inverse, so that U has the dense range which is spanned by (3). This shows that (3) is complete.

In the remainder of our proof, we shall employ a method inspired

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1) Russian Mathematical Survey, **17**, 119–131 (1962). Unfortunately, Bari's original papers are unavailable to the present author.

by the second half of Iséki's proof which is somewhat simpler than that of Birkhoff-Rota.

Let F be the orthocomplement of the space spanned by ψ_{m+1}, $\psi_{m+2},\cdots$ and P the projection belonging to F. To conclude the proof of the theorem, it remains to show that F is m-dimensional. If $f\in F$ is orthogonal to $P\varphi_1, P\varphi_2,\cdots, P\varphi_m$, then we have

$$(f, \varphi_i)=(Pf, \varphi_i)=(f, P\varphi_i)=0, \quad \text{for} \quad i=1, 2,\cdots, m,$$

whence Lemma implies $f=0$ since f is orthogonal to (3). Hence F is spanned by $P\varphi_1, P\varphi_2,\cdots, P\varphi_m$, so that the dimension of F is at most m. On the other hand, the dimension of F is not less than m since F contains $\psi_1, \psi_2,\cdots, \psi_m$. Therefore F is exactly m-dimensional.

In the same manner, the theorem is still true if (1) is replaced by

$$\sum_{n=1}^{\infty}||\varphi_n-\psi_n||<\infty, \tag{1'}$$

which is also due to N. Bari.

References

[1] G. Birkhoff and Gian-Carlo Rota: On the completeness of Sturm-Liouville expansions, Amer. Math. Monthly, **67**, 835-841 (1960).

[2] K. Iséki: On complete orthonormal sets in Hilbert space, Proc. Japan Acad., **33**, 450-452 (1957).

[3] R. Schatten: Norm ideals of completely continuous operators, Ergeb. d. Math. u. Grenzgeb., N. F. Heft 27, Springer, Berlin-Göttingen-Heidelberg (1960).