98. The Equivalence of Two Methods of Absolute Summability

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1.1. Definitions. Let $\sum a_n$ be given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n=p_0+p_1+\cdots+p_n.$$

The sequence-to-sequence transformation:

$$t_n=\frac{\sum_{\nu=0}^{n} p_{n-\nu}s_\nu}{P_n} \quad (P_n\neq 0)$$

defines the sequence $\{t_n\}$ of Nörlund means[1] of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n) to the sum s, if $\lim_{n\to\infty} t_n$ exists and is equal to s, and is said to be absolutely summable $\{N, p_n\}$, or summable $|N, p_n|$,[2] if the sequence $\{t_n\}$ is of bounded variation, that is, the series $\sum|t_n-t_{n-1}|$ is convergent.

In the special case in which $p_n=\dfrac{1}{n+1}$, we write σ_n for t_n, so that

$$\sigma_n=\frac{\sum_{\nu=0}^{n}\dfrac{s_\nu}{n+1-\nu}}{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n+1}}. \tag{1.1.1}$$

We define

$$\sigma_n^*=\frac{1}{\log n}\sum_{\nu=0}^{n}\frac{s_\nu}{n+1-\nu}. \tag{1.1.2}$$

We shall say that $\sum a_n$ is summable H to s, if $\lim_{n\to\infty}\sigma_n$ exists and is equal to s, and that $\sum a_n$ is absolutely summable H or summable $|H|$ if $\{\sigma_n\}$ is of bounded variation. Similarly, if $\lim_{n\to\infty}\sigma_n^*=s$, we shall say that $\sum a_n$ is summable H^*, and if $\{\sigma_n^*\}$ is of bounded variation, we shall say that $\sum a_n$ is absolutely summable H^* or summable $|H^*|$.[3]

1.2. Introduction. It is easy to establish that for ordinary

1) Nörlund [1]; a definition substantially the same as that of Nörlund was given by G. F. Woronoi in the Proceedings of the 11th Congress of Russian naturalists and scientists (in Russian), St. Petersburg, 60–61 (1902). The first English translation of this work of Woronoi with "remarks of the translator" by J. D. Tamarkin is contained in Woronoi [2].

2) Mears [3].

3) Summability H^* is the same as Harmonic Summability as defined by Riesz [4].

summability the methods H and H^* are equivalent,[4] for,

$$\lim_{n\to\infty} \sigma_n^*/\sigma_n = \lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{\log n} = 1.$$

The object of the present paper is to establish the equivalence of the methods of summability $|H|$ and $|H^*|$.

2.1. We establish the following

Theorem. *If $\sum a_n$ is summable $|H|$, it is summable $|H^*|$ and conversely.*

2.2. We require the following lemmas for the proof of our theorem.

Lemma 1. *If $\{\alpha_n\}$ and $\{\beta_n\}$ are any two sequences of bounded variation, then the sequence $\{\alpha_n \cdot \beta_n\}$ is also of bounded variation.*

This result is well known.

Lemma 2.[5] *If* $1+\frac{1}{2}+\cdots+\frac{1}{n} = \log n + \gamma + \varepsilon_n$, *where γ is the Euler's constant, then*

$$\gamma + \varepsilon_n > 0, \textit{ for } n > 1.$$

Lemma 3. *If* $u_n = \dfrac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{\log n}$, *then*

$$\sum_{3}^{\infty} |u_n - u_{n-1}| < \infty,$$

that is, $\{u_n\}$ is a sequence of bounded variation.

Proof. Writing $D_n = \log n \cdot \log(n-1)$,

$$u_n - u_{n-1} = \left[\frac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{\log n} - \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\log(n-1)} \right]$$

$$= D_n^{-1}\left[\left\{ \log n + \log\left(1-\frac{1}{n}\right)\right\}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right) - \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\log n \right]$$

$$= D_n^{-1}\left[\frac{\log n}{n+1} + \left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)\log\left(1-\frac{1}{n}\right)\right]$$

$$= -D_n^{-1}\left[\left(\frac{1}{n}+\frac{1}{2n^2}+\cdots\right)\left(\log n + \varepsilon_n + \gamma + \frac{1}{n+1}\right) - \frac{\log n}{n+1} \right]$$

$$= -D_n^{-1}\left[\left(\frac{1}{n}-\frac{1}{n+1}\right)\log n + \left(\varepsilon_n + \gamma + \frac{1}{n+1}\right)\left(\frac{1}{n}+\frac{1}{2n^2}+\cdots\right)\right.$$

4) Hille and Tamarkin [5], p. 780.
5) Knopp [6], p. 228, Ex. 85a.

$$+\log n\left(\frac{1}{2n^2}+\frac{1}{3n^3}+\cdots\right)\Big]$$

$$<0,$$

by Lemma 2 for $n\geqq 3$.

Hence
$$\sum_3^m |u_n-u_{n-1}|=\sum_3^m(u_{n-1}-u_n)$$
$$=u_2-u_m.$$

Thus
$$\sum_3^\infty |u_n-u_{n-1}|=\frac{1+\frac{1}{2}+\frac{1}{3}}{\log 2}-1,$$

and the Lemma is proved.

Lemma 4. *If $\{u_n\}$ is a sequence of bounded variation and $|u_n|\geq\delta>0$, then $\{1/u_n\}$ is also a sequence of bounded variation.*

The proof is easy.

2.3. Proof of the theorem. Let us write

$$u_n=\frac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{\log n}.$$

We have
$$\sigma^*=\sigma_n u_n.$$

If $\sum a_n$ is summable $|H|$, that is, $\{\sigma_n\}$ is of bounded variation, then by Lemmas 3 and 1, $\{\sigma_n^*\}$ is of bounded variation, or what is the same thing, $\sum a_n$ is summable $|H^*|$.

Conversely, we have,

$$\sigma_n=\frac{\sigma_n^*}{u_n}.$$

Since $\{u_n\}$ is of bounded variation, and u_n is strictly positive, by Lemma 4, $\{1/u_n\}$ is of bounded variation. Hence, if $\{\sigma_n^*\}$ is of bounded variation then $\{\sigma_n\}$ is of bounded variation.

This completes the proof of our theorem.

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